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Motif analysis of noisy regular textures

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Abstract

A regular texture contains a periodic arrangement of a determinable pattern. Motifs of a regular texture are the minimal parallelogrammic sub-regions that can reproduce the texture by tiling. A motif extracted from a regular texture can be used for texture reproduction, texture recognition, and texture compression. However, it is not simple to accurately capture a motif of a regular texture, as the motif may be non-rectangular or arranged in an arbitrary direction. Moreover, when the texture contains additive noises, such as are created by the scanning process, motif analysis becomes more difficult. In this paper, we present an algorithm to obtain a motif of a noisy regular texture in $O(mn \log mn)$ time, where $m \times n$ is the size of the texture. The analysis examples demonstrate that the proposed algorithm can robustly derive motifs of noisy regular textures. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Regular texture; Motif analysis; Translation symmetry; Distance matching function

1. Introduction

A pattern is a general notion that represents an organic structure. A regular texture is a rectangular image which contains the repetition of a visually recognizable unit pattern. Due to structural regularity, a regular texture exhibits an ornamental effect. In commercial areas, such as apparel and interior design, regular textures have been widely used as a fundamental resource to achieve decorative effects (Porter, 1975; Stevens, 1991; Washburn and Crowe, 1992). In computer graphics,

they may be used for texture mapping, which is a low-cost technique to add a realistic appearance to the surface of an object.

A regular texture can be produced from scratch by designing a unit pattern and tiling the pattern in a rectangular region. In this case, however, designing a unit pattern is time-consuming, as it requires not only the original idea but highly developed skills as well. Yet, a variety of sources exist for obtaining regular textures, such as pattern books (Stevens, 1991; Grafton, 1992; Stegenga, 1992). We can derive an image of a regular texture by digitizing with a scanner or a digital camera, and a new texture can be generated by manipulating the scanned image.

Motifs of a regular texture are the minimal sub-regions of the texture which can regenerate the

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texture by tiling. To process a scanned regular texture more effectively, it is necessary to extract a motif from the texture. For example, when we repeat a scanned regular texture to generate a large texture, we are confronted with the problem of visual discontinuity along the boundary (see Figs. 1(a) and (b)). This problem can be avoided by tiling a motif of the texture (see Figs. 1(c) and (d)). In addition, by modifying a motif of a regular texture and tiling the result, we can easily obtain a new regular texture that is similar to the original. However, it is not simple to accurately capture the motif of a regular texture. Extra attention is also required when the motif is either non-rectangular or arranged in an arbitrary direction (see Fig. 2).

A digitized regular texture usually contains additive noises due to the scanning process. Although the noises are almost imperceptible to the human eye, they may destroy the regular structure of the texture in terms of pixel values. That is, the pixel values may not exactly match each other between the repetitions of a visually recognizable motif. Consequently, when a digitized texture *appears* to be regular, it may not be truly regular when we compare pixel values. Thus, it is more difficult to analyze the motif of a digitized regular texture.

In addition to texture production, motif extraction can be used for compression and recognition of regular textures. To save space for storing a regular texture, we can remove redundant information in the texture by retaining only a motif of the texture. When we compare two regular textures, we do not need to compare all pixel information in the textures. Motifs extracted from the textures contain sufficient information for determining the equality of the textures.

In this paper, we propose an algorithm to identify a motif of a regular texture in $O(mn \log mn)$ time, where $m \times n$ is the size of the texture. The algorithm can properly handle a noisy regular texture which preserves a visually regular structure, although its pixel values are not exactly repeated. The analysis examples demonstrate that the algorithm robustly derives the motifs of noisy regular textures.

Symmetry group theory provides a mathematical foundation to define a regular texture and its motif (Armstrong, 1988; Baglivo and Graver, 1983; Connors and Harlow, 1980; Martin, 1982). To analyze the motif of a regular texture, we first adopt the properties of a motif from the theory. To apply the properties to a *noisy* regular texture, we use the distance matching function (Oh et al.,

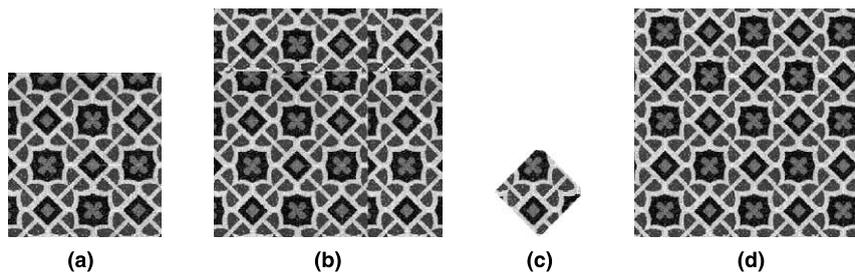


Fig. 1. Repetition of a regular texture: (a) texture; (b) simple repetition; (c) motif; (d) repetition of the motif.

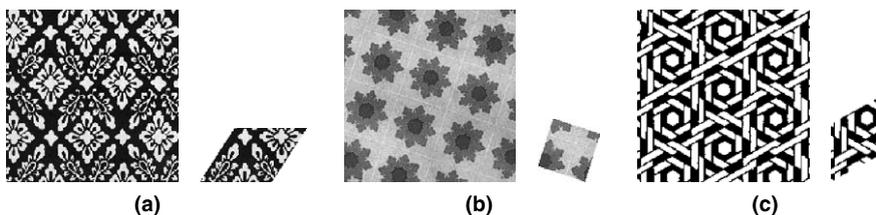


Fig. 2. Regular textures with: (a) non-rectangular; (b) slanted; (c) non-rectangular and slanted motifs.

1999), which can approximate the regular structure of a noisy texture. Several techniques, such as Delaunay triangulation and least-square approximation, are then used to extract a motif by refining possible errors in the approximated regular structure.

The remainder of this paper is organized as follows. Sections 2 and 3 review related work and symmetry group theory, respectively. In Section 4, we present the basic approach of the proposed technique for motif analysis of a noisy regular texture. Sections 5 and 6 explain the details of two steps in the technique: lattice approximation and basis vector approximation. Section 7 gives several analysis examples. We conclude this paper in Section 8.

2. Related work

A texture can be viewed as a two-dimensional string that contains characters from a finite alphabet. Efficient algorithms (Galil and Park, 1992, 1996), which are based on string matching techniques, have been proposed to determine the motif of a regular texture. For a texture of size $m \times n$, the algorithms can determine a motif in $O(mn)$ time. However, the algorithms are complicated and difficult to implement. Furthermore, no mechanism was provided to extract a motif from a noisy regular texture, because the algorithms are based on the equality test of two alphabets.

Several algorithms based on co-occurrence matrices are commonly used to detect the periodicity of a noisy texture (Connors and Harlow, 1980; Parkkinen and Selkainaho, 1990; Zucker and Terzopoulos, 1980). The algorithms first compute co-occurrence matrices for certain displacement vectors and then calculate measures such as χ^2 statistics, κ statistics, and inertias from those matrices. Computing co-occurrence matrices for all possible displacement vectors requires $O(m^2n^2)$ time, where $m \times n$ is the size of the texture. However, the algorithms primarily focus on detecting one-dimensional periodicity and did not address how to extract a motif from the texture, which is related to periodicity in two independent directions.

Oh et al. (1999) proposed the distance matching function to compute the inertias without constructing co-occurrence matrices, which was made possible due to their new interpretation of an inertia. For a texture with the size $m \times n$, the inertias of all co-occurrence matrices can be obtained in $O(mn \log mn)$ time by simultaneously evaluating the distance matching function for all displacement vectors. We use the function to obtain the approximate lattice of a noisy regular texture in Section 5.

3. Preliminary

In this section, we briefly review symmetry group theory, which provides mathematical definitions of a regular texture and its motif (Armstrong, 1988; Baglivo and Graver, 1983; Connors and Harlow, 1980; Martin, 1982).

A texture can be considered as a bivariate function g . The function value $g(x, y)$ represents the gray level at position $(x, y) \in \mathbb{Z}^2$, where \mathbb{Z} is the set of integers. We assume that g is axis-aligned in the two-dimensional plane and $(0, 0)$ is the lower left corner of the texture.

Let \mathbb{R} represent the set of real numbers. A translation τ_v by a vector $\mathbf{v} = (v_x, v_y)$ is a function, $\tau_v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that $\tau_v(\mathbf{p}) = \mathbf{p} + \mathbf{v} = (p_x + v_x, p_y + v_y)$ for a point $\mathbf{p} = (p_x, p_y) \in \mathbb{R}^2$. When a translation τ_v is applied to a texture g , it translates every pixel in g by \mathbf{v} . That is, if g is of size $m \times n$ and we let $g' = \tau_v(g)$, then g' is a texture of the same size as g , and $g'(x, y) = g(x - v_x, y - v_y)$ for $x = v_x, v_x + 1, \dots, v_x + m - 1$ and $y = v_y, v_y + 1, \dots, v_y + n - 1$.

The composition $\tau_v \tau_u$ of two translations τ_u and τ_v is the translation $\tau_{\mathbf{u} + \mathbf{v}}$ by the vector $\mathbf{u} + \mathbf{v}$. For $i, j \in \mathbb{Z}$, $\tau_v^j \tau_u^i$ is the i applications of τ_u followed by j applications of τ_v . Therefore, for a point $\mathbf{p} \in \mathbb{R}^2$, $\tau_v^j \tau_u^i(\mathbf{p}) = \tau_{i\mathbf{u} + j\mathbf{v}}(\mathbf{p}) = \mathbf{p} + i\mathbf{u} + j\mathbf{v}$. Two translations τ_u and τ_v are said to be independent if the two vectors \mathbf{u} and \mathbf{v} are linearly independent.

A texture g is said to have a translation symmetry if there is a translation τ_u such that g is matched with $\tau_u(g)$ in the non-empty overlapping region (Stevens, 1991; Armstrong, 1988; Baglivo and Graver, 1983; Martin, 1982). It is known that

the set of all such translations forms a group under their composition (Armstrong, 1988). The group is called the *translation symmetry group* of g and denoted by \mathcal{T}_g .

A texture g is *regular* if any translation in its translation symmetry group \mathcal{T}_g can be represented by a repeated composition of two independent translations, τ_a and τ_b (Washburn and Crowe, 1992; Armstrong, 1988; Baglivo and Graver, 1983; Martin, 1982; Arnold, 1969). That is, g is regular if two linearly independent vectors \mathbf{a} and \mathbf{b} exist, such that \mathcal{T}_g consists of translations τ_q , where $\mathbf{q} = i\mathbf{a} + j\mathbf{b}$, $i, j \in \mathbb{Z}$. Such a group \mathcal{T}_g shall be denoted by $\mathcal{T}_g = \langle \mathbf{a}, \mathbf{b} \rangle$. Vectors \mathbf{a} and \mathbf{b} are called the *basis vectors* of g . Note that a periodic texture is not regular if it is periodic in only one direction. Only textures that are periodic in two independent directions can be termed regular textures.

For a point $\mathbf{p} \in \mathbb{R}^2$, the lattice of g for \mathbf{p} , denoted by $\mathcal{L}_g(\mathbf{p})$, is the set of all points obtained by applying the translations in \mathcal{T}_g to \mathbf{p} (Armstrong, 1988). That is,

$$\mathcal{L}_g(\mathbf{p}) = \{ \mathbf{q}_{i,j} | \mathbf{q}_{i,j} = \mathbf{p} + i\mathbf{a} + j\mathbf{b}, i, j \in \mathbb{Z} \},$$

where $\mathcal{T}_g = \langle \mathbf{a}, \mathbf{b} \rangle$. When \mathbf{p} is the origin $\mathbf{o} = (0, 0)$, we have

$$\mathcal{L}_g(\mathbf{o}) = \{ \mathbf{q}_{i,j} | \mathbf{q}_{i,j} = i\mathbf{a} + j\mathbf{b}, i, j \in \mathbb{Z} \}.$$

Note that $\mathcal{L}_g(\mathbf{p})$ is just a translation of $\mathcal{L}_g(\mathbf{o})$ by τ_p . In this paper, we consider only the lattice $\mathcal{L}_g(\mathbf{o})$, which is called the *lattice* of g and denoted by \mathcal{L}_g . The origin \mathbf{o} always belongs to \mathcal{L}_g regardless of vectors \mathbf{a} and \mathbf{b} .

Depending on the shape of a parallelogram determined by basis vectors \mathbf{a} and \mathbf{b} , we have five distinct lattice types: parallelogram, rectangle, rhombus, square, and hexagon (Washburn and Crowe, 1992; Armstrong, 1988; Baglivo and Graver, 1983):

- *Parallelogram.* \mathbf{a} and \mathbf{b} do not belong to any of the cases below.
- *Rectangle.* \mathbf{a} and \mathbf{b} have different lengths, and the angle between \mathbf{a} and \mathbf{b} is $\pi/2$.
- *Rhombus.* \mathbf{a} and \mathbf{b} have the same length, and the angle between \mathbf{a} and \mathbf{b} is neither $\pi/3$ nor $\pi/2$.
- *Square.* \mathbf{a} and \mathbf{b} have the same length, and the angle between \mathbf{a} and \mathbf{b} is $\pi/2$.
- *Hexagon.* \mathbf{a} and \mathbf{b} have the same length, and the angle between \mathbf{a} and \mathbf{b} is $\pi/3$.

Fig. 3 shows the configurations of the lattices.

Let g be a regular texture, such that $\mathcal{T}_g = \langle \mathbf{a}, \mathbf{b} \rangle$. A *motif* of g , denoted by \mathcal{M}_g , is a parallelogrammic sub-region of g determined by its origin \mathbf{m} and basis vectors \mathbf{a} and \mathbf{b} . The origin \mathbf{m} of \mathcal{M}_g is a point in g which determines a corner of \mathcal{M}_g . Then, \mathcal{M}_g is specified by the parallelogrammic sub-region of g whose vertices are \mathbf{m} , $\mathbf{m} + \mathbf{a}$, $\mathbf{m} + \mathbf{b}$, and $\mathbf{m} + \mathbf{a} + \mathbf{b}$. See Fig. 4 for an illustration.

A motif \mathcal{M}_g of a regular texture g contains the minimal set of pixels in g such that we can reproduce g by repeating the pixel set. Although basis vectors \mathbf{a} and \mathbf{b} have been determined, the extracted motifs may differ, depending on their origins. However, two motifs with different origins have the same set of pixel information and only the arrangements of pixels differ in the motifs. Any of the motifs can be used to reproduce texture g by tiling. Therefore, to determine a motif \mathcal{M}_g , it is sufficient to identify basis vectors \mathbf{a} and \mathbf{b} . Now, the problem of motif analysis of a regular texture g is reduced to the problem of deriving two vectors \mathbf{a} and \mathbf{b} , such that $\mathcal{T}_g = \langle \mathbf{a}, \mathbf{b} \rangle$.

4. Basic approach

In this section, we first present an approach to analyze a motif of a given regular texture g . We

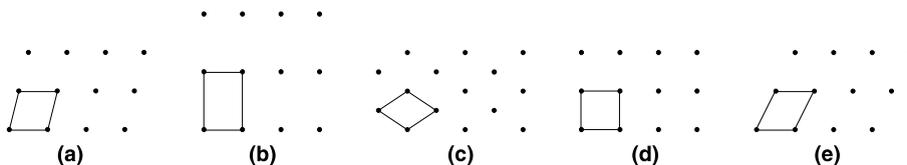


Fig. 3. Five lattice types: (a) parallelogram; (b) rectangle; (c) rhombus; (d) square; (e) hexagon.

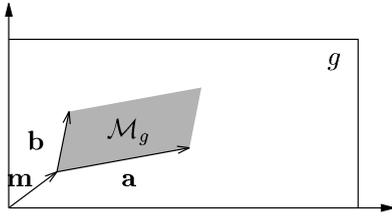


Fig. 4. Motif \mathcal{M}_g of a regular texture g with the origin m .

then extend the approach to handle a noisy regular texture f .

4.1. Motif analysis of a regular texture

Let g be a regular texture, whose lower left corner is positioned at the origin $\mathbf{o} = (0, 0)$. Let \mathbf{a} and \mathbf{b} be basis vectors of g . That is, $\mathcal{T}_g = \langle \mathbf{a}, \mathbf{b} \rangle$. Then, the lattice \mathcal{L}_g of g is given by $\mathcal{L}_g = \{\mathbf{q} | \mathbf{q} = i\mathbf{a} + j\mathbf{b}, i, j \in \mathbb{Z}\}$.

An important property of lattice \mathcal{L}_g is that it can be directly derived from texture g , even when we do not know the basis vectors \mathbf{a} and \mathbf{b} . From the definition of the translation symmetry group, $\tau_{\mathbf{q}}$ is a translation in \mathcal{T}_g for each point \mathbf{q} in \mathcal{L}_g . In other words, g is matched with $\tau_{\mathbf{q}}(g)$ in the overlapping region for any point \mathbf{q} in \mathcal{L}_g . Hence, the lattice \mathcal{L}_g can be obtained by comparing g with its translations and finding all points \mathbf{q} , such that $g = \tau_{\mathbf{q}}(g)$.

After we determine the lattice \mathcal{L}_g , the basis vectors \mathbf{a} and \mathbf{b} can be derived from \mathcal{L}_g . Let \mathbf{q} be a point in \mathcal{L}_g and let $\mathcal{V}_g(\mathbf{q})$ be the set of vectors that connect \mathbf{q} to other points in \mathcal{L}_g . Since the set $\mathcal{V}_g(\mathbf{q})$ is the same regardless of the choice of \mathbf{q} and \mathcal{L}_g always contains the origin \mathbf{o} , we consider the set $\mathcal{V}_g(\mathbf{o})$, which is denoted by \mathcal{V}_g . A pair of vectors \mathbf{a} and \mathbf{b} in \mathcal{V}_g can be basis vectors of g if \mathbf{a} and \mathbf{b} satisfy the following properties (Armstrong, 1988; Galil and Park, 1992):

- Vectors \mathbf{a} and \mathbf{b} are linearly independent.
- The lengths of \mathbf{a} and \mathbf{b} are less than or equal to those of other vectors in \mathcal{V}_g ; that is, $\|\mathbf{a}\| \leq \|\mathbf{v}\|$ and $\|\mathbf{b}\| \leq \|\mathbf{v}\|$ for $\mathbf{v} \in \mathcal{V}_g - \{\mathbf{a}, \mathbf{b}\}$.

Refer to Fig. 3 for an illustration. Unfortunately, such a pair of vectors in \mathcal{V}_g is not unique. For

example, if \mathbf{a} and \mathbf{b} satisfy the properties, so do their negations, $-\mathbf{a}$ and $-\mathbf{b}$.

To uniquely determine the basis vectors \mathbf{a} and \mathbf{b} from \mathcal{V}_g , we first choose \mathbf{a} as the vector in \mathcal{V}_g , such that its angle from the positive x -axis is minimum among the minimum length vectors in \mathcal{V}_g . Note that there are at least two minimum length vectors in \mathcal{V}_g , \mathbf{a} and $-\mathbf{a}$. Then, \mathbf{b} is assigned to the vector in \mathcal{V}_g , such that the angle between \mathbf{a} and \mathbf{b} is minimum among the minimum length vectors in $\mathcal{V}_g - \{\mathbf{a}, -\mathbf{a}\}$. All the angles here are measured in the counterclockwise direction. With this selection scheme, \mathbf{a} is always a vector in the first or second quadrant of the plane and the angle between \mathbf{a} and \mathbf{b} is less than 180° .

Now we can analyze a motif of a given regular texture g in two steps: lattice determination and basis vector determination. In the first step, we obtain the lattice \mathcal{L}_g by searching the points \mathbf{q} that satisfy $g = \tau_{\mathbf{q}}(g)$. The search is performed in the region \mathcal{R}_g , which contains the points \mathbf{p} in \mathbb{R}^2 such that the overlapped region of g and $\tau_{\mathbf{p}}(g)$ is not empty. When the size of g is $m \times n$, we have

$$\mathcal{R}_g = \{(x, y) \mid |x| < m \text{ and } |y| < n\}.$$

In the second step, we derive the vector set \mathcal{V}_g from the lattice \mathcal{L}_g . Then, the basis vector \mathbf{a} is the minimum length vector in \mathcal{V}_g , where we select the vector whose angle from the x -axis is smaller among the vectors with the same length. Similarly, \mathbf{b} is the minimum length vector in $\mathcal{V}_g - \{\mathbf{a}\}$, where we consider the angle measured from \mathbf{a} .

4.2. Motif analysis of a noisy regular texture

Let f be a noisy regular texture, which does not satisfy the definition of a regular texture while still preserving a visually regular structure. Texture f can be regarded as a mixture of a regular texture g and a random noise texture ϵ . That is, $f = g + \epsilon$. The problem of motif analysis of a given noisy regular texture f can be defined as the detection of two basis vectors \mathbf{a} and \mathbf{b} of the underlying regular texture g . Then, a motif \mathcal{M}_f of f can be extracted from a parallelogrammic sub-region of f determined by the vectors \mathbf{a} and \mathbf{b} . By tiling the motif \mathcal{M}_f , we can generate a regular texture f' , which is visually indistinguishable from f .

Let \mathbf{q} be a point in the lattice \mathcal{L}_g of g . Although we have $g = \tau_{\mathbf{q}}(g)$, it may not hold that $f = \tau_{\mathbf{q}}(f)$ due to noise. To handle this problem, we use the distance matching function d_f (Oh et al., 1999), which computes the squared difference between f and $\tau_{\mathbf{p}}(f)$ for a point $\mathbf{p} \in \mathbb{Z}^2$. That is,

$$\begin{aligned} d_f(\mathbf{p}) &= \|f - \tau_{\mathbf{p}}(f)\|^2 \\ &= \|(g - \tau_{\mathbf{p}}(g)) + (\epsilon - \tau_{\mathbf{p}}(\epsilon))\|^2. \end{aligned}$$

For a point \mathbf{q} in \mathcal{L}_g , $d_f(\mathbf{q})$ is reduced to $\|\epsilon - \tau_{\mathbf{q}}(\epsilon)\|^2$. For texture f to preserve visually regular structures, the magnitude of ϵ should be small over the entire domain. This implies that function d_f would have a small value for each point \mathbf{q} in \mathcal{L}_g . Hence, we can approximate lattice \mathcal{L}_g by using the function d_f .

Let \mathcal{L}_f be an approximation of \mathcal{L}_g , which may contain errors. For example, \mathcal{L}_f may contain extra points that do not belong to \mathcal{L}_g . Let \mathcal{V}_f be the set of vectors that connect the origin \mathbf{o} to other points in \mathcal{L}_f . Because of the errors in \mathcal{L}_f , we cannot derive the basis vectors \mathbf{a} and \mathbf{b} of g by simply selecting minimum length vectors from \mathcal{V}_f . For example, extra points in \mathcal{L}_f may introduce vectors to \mathcal{V}_f whose lengths are smaller than those of \mathbf{a} and \mathbf{b} . To resolve this problem, we use the Delaunay triangulation (de Berg et al., 1997) of \mathcal{L}_f , which is the dual of the Voronoi diagram of \mathcal{L}_f . From the edges of the Delaunay triangulation, we can obtain a set of vectors that connect the nearby points in \mathcal{L}_f . The vectors \mathbf{a} and \mathbf{b} are then determined by using the most frequent vectors in the set.

In summary, we analyze a motif of a given noisy regular texture f in two steps: lattice approximation and basis vector approximation. In the first step, the approximate lattice \mathcal{L}_f of f is obtained by using the distance matching function d_f . In the second step, we determine basis vectors \mathbf{a} and \mathbf{b} of f by using the vector set derived from the Delaunay triangulation of \mathcal{L}_f . In Sections 5 and 6, we present the details of the first and second steps, respectively.

5. Lattice approximation

In this section, we first introduce the distance matching function d_f (Oh et al., 1999) of a noisy

regular texture f . Next, we present how to determine the threshold T , where the points \mathbf{p} such that $d_f(\mathbf{p}) \leq T$ are selected. The approximate lattice \mathcal{L}_f is obtained by choosing representative points in the clusters of the selected points. Finally, we explain three types of errors that may exist in \mathcal{L}_f .

5.1. Distance matching function

Let f be a texture of the size $m \times n$. Let $\mathbf{p} = (x, y)$ be a point in \mathbb{Z}^2 . Then, the distance matching function d_f is defined by

$$\begin{aligned} d_f(\mathbf{p}) &= \|f - \tau_{\mathbf{p}}(f)\|^2 \\ &= \begin{cases} d_f^1(x, y), & x \geq 0 \text{ and } y \geq 0, \\ d_f^2(-x, y), & x \leq 0 \text{ and } y \geq 0, \\ d_f^1(-x, -y), & x \leq 0 \text{ and } y \leq 0, \\ d_f^2(x, -y), & x \geq 0 \text{ and } y \leq 0, \end{cases} \end{aligned}$$

where

$$\begin{aligned} d_f^1(x, y) &= \frac{1}{(m-x)(n-y)} \\ &\quad \times \sum_{i=0}^{m-x-1} \sum_{j=0}^{n-y-1} |f(i, j) - f(x+i, y+j)|^2, \\ d_f^2(x, y) &= \frac{1}{(m-x)(n-y)} \\ &\quad \times \sum_{i=0}^{m-x-1} \sum_{j=0}^{n-y-1} |f(m-1-i, j) \\ &\quad \quad - f(m-1-i-x, y+j)|^2. \end{aligned}$$

As shown in Fig. 5, $d_f(\mathbf{p})$ computes the average of squared differences of pixel values over the region where f and $\tau_{\mathbf{p}}(f)$ overlap.

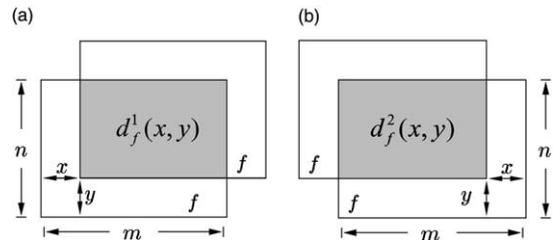


Fig. 5. Schematic diagrams of the distance matching function d_f : (a) $d_f^1(x, y)$; (b) $d_f^2(x, y)$.

The function value $d_f(\mathbf{p})$ is well-defined at points $\mathbf{p} = (x, y)$, $|x| = 0, 1, \dots, m-1$ and $|y| = 0, 1, \dots, n-1$, for which the overlapped region of f and $\tau_p(f)$ is not empty. However, when $|x| > \lfloor m/2 \rfloor$ or $|y| > \lfloor n/2 \rfloor$, the overlapped region is less than half of f , which makes the resulting function values less reliable. Therefore, we only consider the value of d_f at points \mathbf{p} in \mathcal{R}_f , where

$$\mathcal{R}_f = \left\{ (x, y) \mid |x| = 0, 1, \dots, \left\lfloor \frac{m}{2} \right\rfloor \text{ and } |y| = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

By using the algorithm in (Oh et al., 1999), we can compute the function values $d_f(\mathbf{p})$ at all points $\mathbf{p} \in \mathcal{R}_f$ in $O(mn \log mn)$ time.

Fig. 6 shows a noisy regular texture f scanned from a book (Grafton, 1992) and the plot of the corresponding d_f . In Fig. 6(b), the origin \mathbf{o} lies at the center of the image and the brightness at \mathbf{p} is proportional to the function value $d_f(\mathbf{p})$. Note that function values $d_f(\mathbf{p})$ have the same regular structure as texture f .

5.2. Approximate lattice determination

Let f be an $m \times n$ noisy regular texture such that $f = g + \epsilon$, for a regular texture g and a noise ϵ . Since f contains ϵ , the function value $d_f(\mathbf{p})$ may not be zero for a point \mathbf{p} in the lattice \mathcal{L}_g of g . To approximate \mathcal{L}_g , we collect all points \mathbf{q} in \mathcal{R}_f such that $d_f(\mathbf{q})$ is below a given threshold T . That is,

$$\mathcal{L}'_f = \{ \mathbf{q} \mid d_f(\mathbf{q}) \leq T, \mathbf{q} \in \mathcal{R}_f \}.$$

If the threshold T is large, every point in \mathcal{L}_g could be included in \mathcal{L}'_f , but we may have many extra points in \mathcal{L}'_f which do not belong to \mathcal{L}_g .

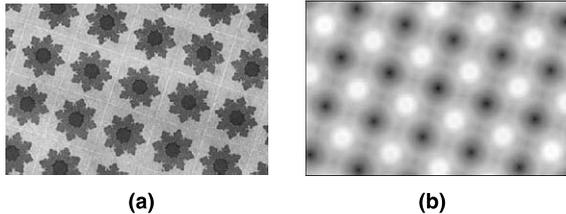


Fig. 6. A texture f and its distance matching function d_f : (a) texture f ; (b) the plot of d_f .

When we select a smaller T , the number of extra points in \mathcal{L}'_f decreases, but \mathcal{L}'_f can miss points in \mathcal{L}_g .

To obtain a reasonable threshold T , we consider the expected value μ_d of $d_f(\mathbf{p})$ for an arbitrary point $\mathbf{p} = (x, y)$ in \mathcal{R}_f . We may assume that the pixel values of f are integers between 0 and Q . Let h be the histogram of f , where $h(c)$ gives the number of pixels in f whose values are c . Consider a pixel $\mathbf{q} = (i, j)$ in f and let c represent its pixel value. Pixel \mathbf{q} is compared to the pixel $\mathbf{q}' = (i+x, j+y)$ in computing $d_f(\mathbf{p})$. The probability that pixel value $f(\mathbf{q}')$ is d can be approximated by $h(d)/mn$. Hence, the expected squared difference between pixel values of \mathbf{q} and \mathbf{q}' is $\sum_{d=0}^Q (h(d)/mn)(c-d)^2$. Then, the expected value μ_d of $d_f(\mathbf{p})$ can be obtained by weighted averaging these expected squared differences over all pixel values c . That is,

$$\begin{aligned} \mu_d &= \frac{1}{mn} \sum_{c=0}^Q h(c) \sum_{d=0}^Q \frac{h(d)}{mn} (c-d)^2 \\ &= \frac{1}{m^2 n^2} \sum_{c=0}^Q \sum_{d=0}^Q h(c)h(d)(c-d)^2. \end{aligned}$$

For a point \mathbf{q} in \mathcal{L}_g , function value $d_f(\mathbf{q})$ would be much less than μ_d , which is the expected value of $d_f(\mathbf{p})$ for an arbitrary point \mathbf{p} in \mathcal{R}_f . We introduce the noisiness parameter α between 0 and 1, and use $\alpha\mu_d$ as the threshold T to obtain \mathcal{L}'_f . When f appears noisy, we use a larger α , so that \mathcal{L}'_f contains as many points in \mathcal{L}_g as possible. A smaller α would be sufficient for a less noisy f . In our experiments, α between 0.2 and 0.5 worked well for most noisy regular textures.

When a relatively large threshold T is used, the point set \mathcal{L}'_f may contain many extra points that do not reside in \mathcal{L}_g . For example, Fig. 7(a) shows the plot of \mathcal{L}'_f obtained by applying the threshold T with $\alpha = 0.5$ to the function values $d_f(\mathbf{p})$ in Fig. 6(b). As we can see in Fig. 7(a), the points in \mathcal{L}'_f tend to form clusters around the points in \mathcal{L}_g . To refine \mathcal{L}'_f , we decompose \mathcal{L}'_f into point clusters and select a representative point from each cluster.

Let \mathcal{S}_i , $i = 0, 1, \dots, l-1$, represent the point clusters in \mathcal{L}'_f , which satisfy the following conditions:

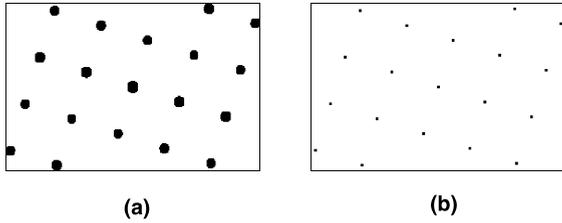


Fig. 7. Example of lattice approximation: (a) point set \mathcal{L}'_f ; (b) approximate lattice \mathcal{L}_f .

- $\mathcal{L}'_f = \bigcup_{i=0}^{l-1} \mathcal{S}_i$ and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ for all $i \neq j$.
- For all $\mathbf{p}, \mathbf{q} \in \mathcal{S}_i$, there exists an 8-connected path in \mathcal{S}_i from \mathbf{p} to \mathbf{q} .

The first condition implies that the set of clusters \mathcal{S}_i is a partition of \mathcal{L}'_f . With the second condition, each cluster \mathcal{S}_i is an 8-connected component of \mathcal{L}'_f . In each cluster \mathcal{S}_i , we choose a point \mathbf{q}_i whose function value $d_f(\mathbf{q}_i)$ is the minimum among those of the points in \mathcal{S}_i . We can consider the set of such points \mathbf{q}_i as an approximation of the lattice \mathcal{L}_g . The point set is denoted by \mathcal{L}_f and called the *approximate lattice* of f :

$$\mathcal{L}_f = \{\mathbf{q}_i | d_f(\mathbf{q}_i) \leq d_f(\mathbf{q}) \text{ for all } \mathbf{q} \in \mathcal{S}_i, \\ i = 0, 1, \dots, l-1\}.$$

For a given noisy regular texture f of size $m \times n$, the approximate lattice \mathcal{L}_f can be obtained in $O(mn \log mn)$ time, because the evaluation of the distance matching function d_f dominates the computation time. Fig. 7(b) shows the plot of \mathcal{L}_f derived from \mathcal{L}'_f shown in Fig. 7(a). From Fig. 7(b), we observe that the clustering process effectively eliminates extra points in \mathcal{L}'_f .

5.3. Possible errors in approximate lattice \mathcal{L}_f

Let f be a noisy regular texture and g the underlying regular texture. The approximate lattice \mathcal{L}_f of f may be different from the lattice \mathcal{L}_g of g . This is due to three types of errors: perturbation errors, missing points, and extra points. With a perturbation error, the position of a point \mathbf{q} in \mathcal{L}_g changes slightly in \mathcal{L}_f . This error occurs when point \mathbf{q} does not have the minimum value of function d_f among the points in the cluster to which it belongs. A point \mathbf{q} in \mathcal{L}_g can be missed in

\mathcal{L}_f if there is much noise in the overlapping region of f and $\tau_q(f)$, so that $d_f(\mathbf{q})$ is greater than the threshold T . An extra point \mathbf{q} may exist in \mathcal{L}_f , which is not contained in \mathcal{L}_g , if the noise in the overlapping region of f and $\tau_q(f)$ accidentally reduces $d_f(\mathbf{q})$ to below T .

Among these three types of errors, we can almost entirely eliminate the missing points by taking a relatively large threshold T . In the experiments, threshold $T = \alpha \mu_d$ with $\alpha = 0.5$ usually resulted in approximate lattices \mathcal{L}_f with no missing points. In the next section, we present a robust method to determine the basis vectors \mathbf{a} and \mathbf{b} of g from \mathcal{L}_f , which properly handles perturbation errors and extra points.

6. Basis vector approximation

In this section, we first investigate the property of the Delaunay triangulation of a lattice \mathcal{L}_g of a regular texture g . Then, the property is applied to the approximate lattice \mathcal{L}_f of a noisy regular texture f to determine the basis vectors \mathbf{a} and \mathbf{b} of f .

6.1. Delaunay triangulation of a lattice

Let g be a regular texture. As mentioned in Section 3, the lattice \mathcal{L}_g of g belongs to one of the five lattice types; parallelogram, rectangle, rhombus, square, and hexagon. Fig. 8 shows the Delaunay triangulations of \mathcal{L}_g for the five lattice types. Note that in the Delaunay triangulation of a point set, there are edges only between nearby points (de Berg et al., 1997).

Let \mathcal{D}_g be the Delaunay triangulation of \mathcal{L}_g . Regardless of the lattice type of \mathcal{L}_g , there are six vectors in \mathcal{D}_g which connect the origin \mathbf{o} to its neighboring points in \mathcal{L}_g (see Fig. 8). Recall that \mathbf{o} always belong to \mathcal{L}_g . These vectors can be represented by three linearly independent vectors, \mathbf{c}_1 , \mathbf{c}_2 , and \mathbf{c}_3 , and their negations. Without loss of generality, we may assume that $0^\circ \leq \angle \mathbf{c}_1 < \angle \mathbf{c}_2 < \angle \mathbf{c}_3 < 180^\circ$, where $\angle \mathbf{c}_i$ denotes the angle of \mathbf{c}_i measured from the positive x -axis. As we can see in Fig. 8, the vectors \mathbf{c}_i satisfy that $\mathbf{c}_3 = \mathbf{c}_2 - \mathbf{c}_1$ for all lattice types.

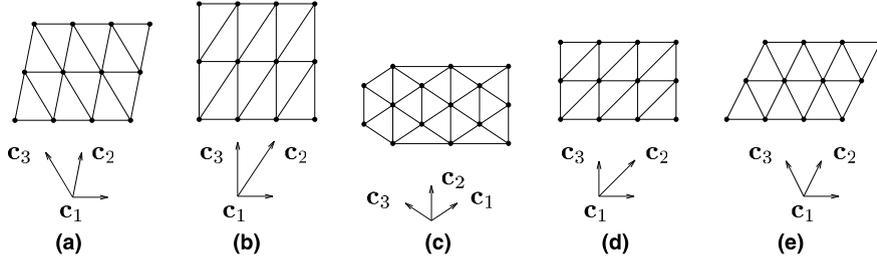


Fig. 8. Delaunay triangulations for five lattice types: (a) parallelogram; (b) rectangle; (c) rhombus; (d) square; (e) hexagon.

Let \mathbf{a} and \mathbf{b} represent two vectors among \mathbf{c}_1 , \mathbf{c}_2 , and \mathbf{c}_3 , such that $\|\mathbf{a}\| \leq \|\mathbf{c}_i\|$ for all \mathbf{c}_i and $\|\mathbf{b}\| \leq \|\mathbf{c}_i\|$ for $\mathbf{c}_i \neq \mathbf{a}$. Then, \mathbf{a} and \mathbf{b} satisfy the property of the basis vectors of g specified in Section 4.1. That is, \mathbf{a} and \mathbf{b} are linearly independent, and their lengths are less than or equal to those of other vectors in \mathcal{V}_g , where \mathcal{V}_g is the set of vectors from \mathbf{o} to other points in \mathcal{L}_g . This implies that we are not required to consider all vectors in \mathcal{V}_g to determine the basis vectors of g . Instead, it is sufficient to consider only six vectors in $\mathcal{C} = \{\pm\mathbf{c}_1, \pm\mathbf{c}_2, \pm\mathbf{c}_3\}$. Based on the selection scheme outlined in Section 4.1, we choose the basis vectors \mathbf{a} and \mathbf{b} of g as the minimum length vectors in \mathcal{C} and $\mathcal{C} - \{\mathbf{a}, -\mathbf{a}\}$ whose angles from the x -axis and \mathbf{a} are the smallest, respectively.

Although the vectors \mathbf{c}_i are obtained from the edges in \mathcal{D}_g which connect \mathbf{o} to its neighbor points, they can also be used to represent the other edges in \mathcal{D}_g . Since \mathcal{D}_g has a regular structure, for any interior point \mathbf{q} in \mathcal{D}_g , an edge from \mathbf{q} to its nearby point corresponds to one of the six vectors in \mathcal{C} (see Fig. 8). That is, the set of all interior edges in \mathcal{D}_g can be reduced to the vector set \mathcal{C} . We use this property to determine the three vectors, \mathbf{c}_1 , \mathbf{c}_2 , and \mathbf{c}_3 , from the approximate lattice \mathcal{L}_f of a noisy regular texture f .

6.2. Basis vector determination

Let f be a noisy regular texture, such that $f = g + \epsilon$, for a regular texture g and a noise ϵ . Let \mathcal{D}_f be the Delaunay triangulation of the approximate lattice \mathcal{L}_f . Due to the errors in \mathcal{L}_f , the set of interior edges in \mathcal{D}_f may not be represented by three vectors, \mathbf{c}_1 , \mathbf{c}_2 , and \mathbf{c}_3 . For example, an extra

point in \mathcal{L}_f introduces an edge to \mathcal{D}_f which has no relation with \mathbf{c}_i . Also, with the perturbation errors in \mathcal{L}_f , slightly different vectors are extracted from the edges in \mathcal{D}_f for which would result in the same vector \mathbf{c}_i if there were no perturbation errors. Hence, we determine \mathbf{c}_1 , \mathbf{c}_2 , and \mathbf{c}_3 by finding the three largest clusters in the vectors obtained from the edges in \mathcal{D}_f and choosing representative vectors in the clusters.

Let e be an edge in \mathcal{D}_f that connects two points \mathbf{q} and \mathbf{r} . Without loss of generality, we assume that $q_x < r_x$ if $q_y = r_y$, and $q_y < r_y$ otherwise, where $\mathbf{q} = (q_x, q_y)$ and $\mathbf{r} = (r_x, r_y)$. Then, edge e is considered as the vector from \mathbf{q} to \mathbf{r} and represented by the point $\mathbf{r} - \mathbf{q}$ in the plane. With this conversion, each edge in \mathcal{D}_f can be mapped to a point in the first or second quadrant of the plane.

Let \mathcal{E}_f be the set of such points derived from the interior edges in \mathcal{D}_f . We decompose \mathcal{E}_f into a set of point clusters \mathcal{S}_i by the same method we performed for \mathcal{L}'_f in Section 5.2. Next, we choose the three largest clusters from \mathcal{S}_i in terms of the number of points in each cluster. By averaging the points in each of these three clusters, we obtain three vectors \mathbf{c}'_1 , \mathbf{c}'_2 , and \mathbf{c}'_3 , where $0^\circ \leq \angle \mathbf{c}'_1 < \angle \mathbf{c}'_2 < \angle \mathbf{c}'_3 < 180^\circ$.

Since lattice \mathcal{L}_f may contain perturbation errors, the vectors \mathbf{c}'_i do not necessarily satisfy $\mathbf{c}'_3 = \mathbf{c}'_2 - \mathbf{c}'_1$. We determine the vectors \mathbf{c}_1 , \mathbf{c}_2 , and \mathbf{c}_3 from \mathbf{c}'_i by minimizing the least-square error

$$\text{err}(\mathbf{c}_1, \mathbf{c}_2) = \frac{1}{2} (\|\mathbf{c}_1 - \mathbf{c}'_1\|^2 + \|\mathbf{c}_2 - \mathbf{c}'_2\|^2 + \|(\mathbf{c}_2 - \mathbf{c}_1) - \mathbf{c}'_3\|^2).$$

Vectors \mathbf{c}_1 and \mathbf{c}_2 that minimize $\text{err}(\mathbf{c}_1, \mathbf{c}_2)$ can be obtained by solving the linear system of equations

$$\frac{\partial \text{err}}{\partial \mathbf{c}_1} = 2\mathbf{c}_1 - \mathbf{c}_2 - (\mathbf{c}'_1 - \mathbf{c}'_3) = \mathbf{0},$$

$$\frac{\partial \text{err}}{\partial \mathbf{c}_2} = -\mathbf{c}_1 + 2\mathbf{c}_2 - (\mathbf{c}'_2 + \mathbf{c}'_3) = \mathbf{0}.$$

From the equations, we finally obtain

$$\mathbf{c}_1 = \frac{2\mathbf{c}'_1 + \mathbf{c}'_2 - \mathbf{c}'_3}{3}, \quad \mathbf{c}_2 = \frac{\mathbf{c}'_1 + 2\mathbf{c}'_2 + \mathbf{c}'_3}{3},$$

$$\mathbf{c}_3 = \mathbf{c}_2 - \mathbf{c}_1.$$

As mentioned in Section 5.3, lattice \mathcal{L}_f may contain extra points and perturbation errors if we employed a large threshold T to obtain the point set \mathcal{L}'_f . When we determine the vectors \mathbf{c}_i from \mathcal{L}'_f with the above approach, the effects of the extra points are properly removed by selecting the three largest clusters from the point set \mathcal{E}_f . The perturbation errors are handled by averaging the points in the selected clusters. The least-square fitting of \mathbf{c}_1 and \mathbf{c}_2 further enhances the robustness of the method against perturbation errors.

The Delaunay triangulation of a point set with p points can be constructed in $O(p \log p)$ time (de Berg et al., 1997). Let $m \times n$ be the size of a given noisy regular texture f . Then, lattice \mathcal{L}_f contains at most mn points and the Delaunay triangulation \mathcal{D}_f of \mathcal{L}_f can be obtained in $O(mn \log mn)$ time. This implies that we can obtain the three vectors, \mathbf{c}_1 , \mathbf{c}_2 , and \mathbf{c}_3 , from \mathcal{L}_f in $O(mn \log mn)$ time because the construction of \mathcal{D}_f dominates the computation time. The basis vectors \mathbf{a} and \mathbf{b} of f can be determined from $\mathcal{C} = \{\pm \mathbf{c}_1, \pm \mathbf{c}_2, \pm \mathbf{c}_3\}$, as mentioned in Section 6.1.

7. Motif analysis examples

In this section, we first explain in detail how to extract a motif \mathcal{M}_f from a noisy regular texture f when we know the basis vectors \mathbf{a} and \mathbf{b} of f . We

then give several examples of motif analysis of noisy regular textures.

7.1. Motif extraction

Let f be a noisy regular texture and \mathbf{a} and \mathbf{b} be the basis vectors of f determined by the techniques proposed in Sections 5 and 6. To obtain a motif \mathcal{M}_f of f by using \mathbf{a} and \mathbf{b} , we must select the origin \mathbf{m} of \mathcal{M}_f among the points in f . Although other choices are possible, we select $\mathbf{m} = (m_x, m_y)$ such that $m_x = -\min\{0, a_x, b_x, a_x + b_x\}$ and $m_y = -\min\{0, b_y\}$, where $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$. Recall that a_x , b_x , and b_y may have negative values. With this choice of \mathbf{m} , the motif \mathcal{M}_f , which is the parallelogrammic sub-region determined by \mathbf{m} , \mathbf{a} , \mathbf{b} , and $\mathbf{a} + \mathbf{b}$, is guaranteed to be contained in f .

After origin \mathbf{m} has been selected, motif \mathcal{M}_f can be generated by collecting pixels \mathbf{p} of f in the parallelogrammic sub-region. That is,

$$\mathcal{M}_f(\mathbf{p}) = f(\mathbf{m} + \mathbf{p}) \text{ for } \mathbf{p} = k_1\mathbf{a} + k_2\mathbf{b},$$

where $0 \leq k_1, k_2 < 1$.

By tiling \mathcal{M}_f in the directions of \mathbf{a} and \mathbf{b} , we obtain a regular texture f' of an arbitrary size, which is visually indistinguishable from f .

7.2. Examples

Fig. 9(b) shows the analyzed motif of the texture in Fig. 9(a), which is the same as in Fig. 6(a). By tiling the motif, we obtain the texture in Fig. 9(c), which is larger than the original. The texture in Fig. 9(d) was scanned from a book (Hornung, 1976). Its motif and a texture generated by tiling the motif are shown in Figs. 9(e) and (f), respectively. The texture in Fig. 9(g) was obtained from a web site (Schofield, 2001) and is commonly referred as the *Escher's design*. Figs. 9(h) and (i) show its motif and a texture generated by tiling the motif, respectively. The texture in Fig. 9(j) was obtained by scanning a texture in a book (Stegenga, 1992). Figs. 9(k) and (l) show a motif and a texture generated by tiling the motif, respectively.

For a given $m \times n$ noisy regular texture, lattice and basis vector approximations can be performed

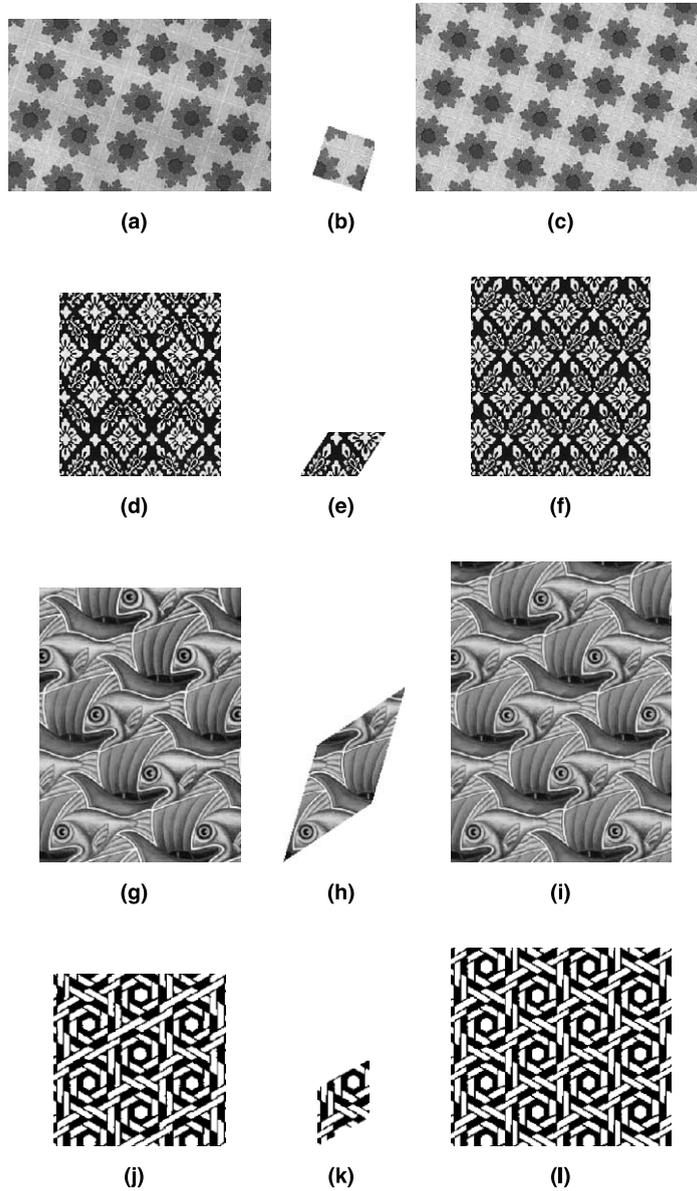


Fig. 9. Motif analysis examples of noisy regular textures.

in $O(mn \log mn)$ time, as described in Sections 5 and 6, respectively. Hence, we can obtain the basis vectors \mathbf{a} and \mathbf{b} from f in $O(mn \log mn)$ time. The sizes of the example textures in Fig. 9, from top to bottom, are 273×179 , 210×235 , 209×284 , and 208×187 , respectively. For the textures, the computation times for the motifs shown in Fig. 9 are

9, 5, 12, and 6 seconds on an SGI Indigo2 with a R10000 processor, respectively.

8. Conclusion

A regular texture contains spatial redundancy due to its translation symmetry. By contrast, a

motif of the texture avoids such redundancy and only includes a minimal pattern that can reproduce the texture by tiling. In this respect, a motif gives a more compact representation of a regular texture without losing structural information.

In this paper, we proposed an algorithm to obtain a motif of a noisy regular texture in $O(mn \log mn)$ time, where $m \times n$ is the size of the texture. This algorithm is applicable to texture analysis, texture recognition, and texture compression.

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References

- Armstrong, M.A., 1988. *Groups and Symmetry*. Springer, New York.
- Arnold, F., 1969. *The Geometry of Regular Repeating Pattern*. The Textile Institute.
- Baglivo, J.A., Graver, J.E., 1983. *Incidence and Symmetry in Design and Architecture*. Cambridge University Press, Cambridge, MA.
- Connors, R.W., Harlow, C.A., 1980. Toward a structural analyzer based on statistical methods. *Comput. Graphics, Image Process.* 12, 224–256.
- de Berg, M., van Kreveld, M., Overmars, M., Schwarzkopf, O., 1997. *Computational Geometry: Algorithms and Applications*. Springer, Berlin.
- Galil, Z., Park, K., 1992. Truly alphabet-independent two-dimensional pattern matching. In: *Symposium on Foundations of Computer Science FOCS*, pp. 247–256.
- Galil, Z., Park, K., 1996. Alphabet-independent two-dimensional witness computation. *SIAM J. Comput.* 25 (5), 907–935.
- Grafton, C.B., 1992. *Decorative Tile Designs*. Dover Publications, New York.
- Hornung, C.P., 1976. *All over Patterns for Designers and Craftsmen*. Dover Publications, New York.
- Martin, G.E., 1982. *Transformation Geometry*. Springer, Berlin.
- Oh, G., Lee, S., Shin, S.Y., 1999. Fast determination of textural periodicity using distance matching function. *Pattern Recognition Lett.* 20 (2), 191–197.
- Parkkinen, J., Selkainaho, K., 1990. Detecting texture periodicity from the co-occurrence matrix. *Pattern Recognition Lett.* 11, 43–50.
- Porter, A.W., 1975. *Principles of Design Pattern*. Davis Publications Inc.
- Schofield, P., 2001. Available from <http://www.worldofescher.com>.
- Stegenga, W., 1992. *Pictorial Archive of Geometric Designs*. Dover Publications, New York.
- Stevens, P.S., 1991. *Handbook of Regular Patterns – An Introduction to Symmetry in Two Dimensions*. MIT Press, Cambridge, MA.
- Washburn, D.K., Crowe, D.W., 1992. *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*. University of Washington Press.
- Zucker, S.W., Terzopoulos, D., 1980. Finding structure in co-occurrence matrices for texture analysis. *Comput. Graphics, Image Process.* 12, 286–308.