Local Injectivity Conditions of 2D and 3D Uniform Cubic B-spline Functions

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Abstract

Uniform cubic B-spline functions have been used for mapping functions in various areas such as image warping and morphing, 3D deformation, and volume morphing. The injectivity (one-to-one property) of a mapping function is important to obtain good results in these areas. This paper considers the local injectivity conditions of 2D and 3D uniform cubic B-spline functions. We propose a geometric interpretation of the local injectivity of a uniform cubic B-spline function, with which 2D and 3D cases can be handled in a similar way. Based on the geometric interpretation, we present sufficient conditions for the local injectivity that are represented in terms of control point displacements. These sufficient conditions are simple and easy to check and will be useful to guarantee the injectivity of mapping functions in application areas.

1. Introduction

Mapping functions that transform certain domains into themselves are widely used in computer graphics. In image warping and morphing, an image is distorted by a 2D mapping function that gives a new position for each point in the image [14]. In deformation techniques such as free-form deformations [12], 3D mapping functions are used to determine the deformed positions of object points. In volume morphing, user-specified features are aligned by distorting given volumes with 3D mapping functions [2, 10].

In these areas, the injectivity (one-to-one property) of a mapping function is important in order to obtain good results. In image warping and morphing, if a mapping function is not injective, the resulting distorted image may contain undesirable wrinkles because parts of the original image fold upon nearby parts. Several techniques have been developed to generate injective mapping functions for image warping and morphing [3, 7, 8, 9]. In deformation techniques, the injectivity of a mapping function guarantees that no self-intersection is introduced to an object in the deformation process. In volume morphing, there is no ambiguity in determining the voxel values of a distorted volume if the mapping function is injective.

Due to their local control property and simplicity, uniform cubic B-spline functions have been used for mapping functions in image morphing and 3D deformation. A 2D uniform cubic B-spline function is defined by applying uniform cubic B-spline bases to 2D control points in a 2D control lattice. Similarly, a 3D uniform cubic B-spline function is determined by a 3D parallelepiped control lattice that consists of 3D control points. Note that each of the 2D and 3D B-spline functions is different from a B-spline surface, which is a function from 2D to 3D obtained by a 2D control lattice with 3D control points. Lee *et al*. used 2D B-spline functions to efficiently generate mapping functions in image morphing [8, 9]. 3D B-spline functions have been adopted to develop direct manipulation techniques for free-form deformations [5, 6].

To obtain injective mapping functions in image morphing, Lee *et al*. presented a sufficient condition for the injectivity of a 2D uniform cubic B-spline function [8, 9]. The sufficient condition provides a single bound for the displacements of control points that guarantees the injectivity of a 2D B-spline function. However, this condition cannot handle the case in which some of control point displacements are above the bound and the others are far below the bound but the resulting function is still injective. Goodman and Unsworth proposed a sufficient condition for a 2D Bézier surface to be injective [4], which can be applied to a 2D B-spline function. For an $m \times n$ lattice of control points, the condition contains $2m(m+1) + 2n(n+1)$ linear inequalities. Unfortunately, when the number of control points is large, the time to check the condition becomes pro-

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hibitive. Although it could be useful in 3D deformation and volume morphing, there has been no research on an injectivity condition for a 3D B-spline function.

In this paper, we consider the injectivity conditions of 2D and 3D uniform cubic B-spline functions. We first propose a geometric interpretation of the injectivity of a 2D B-spline function. Based on this geometric interpretation, we obtain novel sufficient conditions for the injectivity of a 2D B-spline function. The sufficient conditions are represented by inequalities of control point displacements and cover more cases than the previous result [8, 9]. To examine the injectivity condition of a 3D B-spline function, which has not been explored yet, we expand the geometric interpretation of injectivity to 3D. Sufficient conditions for the injectivity of a 3D B-spline function are then obtained, which are also represented by inequalities of control point displacements.

The remainder of this paper is organized as follows. In Section 2, we review the mathematical preliminaries. Sections 3 and 4 consider the injectivity conditions of 2D and 3D B-spline functions, respectively. Section 5 concludes this paper with future work.

2. Mathematical Preliminaries

Let F_2 be a 2D uniform cubic B-spline function defined with an $(m+3)\times(n+3)$ control lattice Φ. Function F_2 consists of $m \times n$ 2D patches, each of which is determined by 4×4 control points in R^2. The injectivity of function F_2 may be violated in two cases. First, a global violation of the injectivity happens when a part in a patch of F_2 intersects a separate part in the same patch or another patch. Due to the convex hull property of B-splines, this global violation is possible only if control lattice Φ contains a self-intersection. Second, the injectivity may be violated locally among connected parts in a patch even though control lattice Φ is not self-intersecting. Although it is counterintuitive that a B-spline function may not be injective when a control lattice does not self-intersect, Lee *et al*. have shown an example of such configuration [9].

In applications of B-spline functions such as morphing and deformation, the global violation of injectivity is not allowed in most cases and can be prevented by using self-intersection-free control lattices. Hence, in this paper, we focus on the local injectivity of a B-spline function, which cannot be determined by checking the self-intersection of a control lattice.

When we investigate sufficient conditions for the local injectivity of function F_2, it is sufficient to consider only one patch of F_2 because the same conditions can be applied to all other patches. Without loss of generality, we represent a 2D uniform cubic B-spline function by a patch f_2, which is defined by

$$\begin{aligned} f_2(u,v) &= (x,y) \qquad (1)\\ &= \sum_{i=0}^{3}\sum_{j=0}^{3} B_i(u)B_j(v)\phi_{ij}, \end{aligned}$$

where $0 \le u, v \le 1$. Uniform cubic B-spline basis functions, B_0, B_1, B_2, and B_3, are defined by

$$\begin{aligned} B_0(u) &= \frac{(1-u)^3}{6},\\ B_1(u) &= \frac{3u^3-6u^2+4}{6},\\ B_2(u) &= \frac{-3u^3+3u^2+3u+1}{6},\\ B_3(u) &= \frac{u^3}{6}, \end{aligned}$$

where $0 \le u \le 1$. $\phi_{ij} = (x_{ij}, y_{ij})$, $i, j = 0, 1, 2, 3$, are 4×4 control points that determine function f_2. Similarly, we represent a 3D uniform cubic B-spline function by a patch f_3, which is defined by

$$\begin{aligned} f_3(u,v,w) &= (x,y,z) \qquad (2)\\ &= \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} B_i(u)B_j(v)B_k(w)\phi_{ijk}, \end{aligned}$$

where $0 \le u, v, w \le 1$.

Function f_2 is locally injective if and only if its Jacobian matrix is nonsingular all over the domain [1]. The Jacobian matrix of f_2 is defined by

$$J(f_2) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}.$$

It is known that a square matrix is nonsingular if and only if its row vectors are linearly independent [13]. Hence, function f_2 is locally injective if and only if two 2D vectors, $(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v})$ and $(\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v})$, are linearly independent. The Jacobian matrix of function f_3 is defined by

$$J(f_3) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

Function f_3 is locally injective if and only if three 3D vectors, $(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial x}{\partial w})$, $(\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial y}{\partial w})$, and $(\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w})$, are linearly independent. In Sections 3 and 4, these conditions are interpreted geometrically to obtain sufficient conditions for the local injectivity of functions f_2 and f_3, respectively.

The following properties of uniform cubic B-spline basis functions and their derivatives are used in later sections.

- $B_i(u) \geq 0$, for $i = 0, 1, 2, 3$
- $\sum_{i=0}^{3} B_i(u) = 1$
- $B_i(u) = B_{3-i}(1-u)$, for $i = 0, 1, 2, 3$
- $\sum_{i=0}^{3} |B'_i(u)| = -2u^2 + 2u + 1 \leq \frac{3}{2}$
- $B'_i(u) = -B'_{3-i}(1-u)$, for $i = 0, 1, 2, 3$

3. Local Injectivity Conditions of 2D B-spline Functions

In this section, we first propose a geometric interpretation of the local injectivity condition of a 2D uniform cubic B-spline function f_2. The two row vectors in the Jacobian matrix $J(f_2)$ are mapped to two regions in R^2 such that f_2 is locally injective if no line simultaneously passes through the origin and the two regions. We then obtain sufficient conditions for the local injectivity of f_2 by computing control point displacements that guarantee such a configuration of the two regions.

3.1. Geometric interpretation of 2D local injectivity

Let f_2 be a 2D uniform cubic B-spline function defined by Eq. (1). The injectivity of function f_2 is determined by the configuration of the 4×4 control points ϕ_{ij}, $i, j = 0, 1, 2, 3$. When ϕ_{ij} equals $\phi^0_{ij} = (i-1, j-1)$ for all i, j, function f_2 is reduced to an identity function. Let $\Delta\phi_{ij}$ be the displacement of control point ϕ_{ij} from ϕ^0_{ij}, that is, $\Delta\phi_{ij} = \phi_{ij} - \phi^0_{ij} = (\Delta x_{ij}, \Delta y_{ij})$. Then, function f_2 can be represented by

$$f_2(u,v) = (u,v) + \sum_{i=0}^{3}\sum_{j=0}^{3} B_i(u)B_j(v)\Delta\phi_{ij}.$$

Now we have

$$\frac{\partial x}{\partial u} = 1 + \sum_{i=0}^{3}\sum_{j=0}^{3} D^u_{ij}(u,v)\Delta x_{ij},$$

$$\frac{\partial x}{\partial v} = \sum_{i=0}^{3}\sum_{j=0}^{3} D^v_{ij}(u,v)\Delta x_{ij},$$

$$\frac{\partial y}{\partial u} = \sum_{i=0}^{3}\sum_{j=0}^{3} D^u_{ij}(u,v)\Delta y_{ij},$$

$$\frac{\partial y}{\partial v} = 1 + \sum_{i=0}^{3}\sum_{j=0}^{3} D^v_{ij}(u,v)\Delta y_{ij},$$

where $D^u_{ij}(u,v) = B'_i(u)B_j(v)$ and $D^v_{ij}(u,v) = B_i(u)B'_j(v)$.

Let Ω_2 be a rectangular domain in R^2 that contains points (u,v) such that $0 \leq u, v \leq 1$. Let $r_1 = (\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v})$ and $r_2 = (\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v})$ denote the row vectors of Jacobian matrix $J(f_2)$. Note that r_1 and r_2 are functions from Ω_2 to R^2 that depend on u and v. Function f_2 is locally injective all over the domain Ω_2 if and only if vectors r_1 and r_2 are linearly independent for each (u,v) in Ω_2.

A 2D vector (x,y) can be interpreted as a point (x,y) in R^2. In the following, we interchangeably use a 2D vector and a point in R^2. Two different 2D vectors (x_1,y_1) and (x_2,y_2) are linearly independent if and only if the line passing through the two points (x_1,y_1) and (x_2,y_2) does not intersect the origin. Hence, function f_2 is locally injective over Ω_2 if and only if no line simultaneously passes through the origin, r_1, and r_2 for any (u,v) in Ω_2.

Let $S_2(c,\delta)$ be a region in R^2 defined by

$$S_2(c,\delta) = \{(x + c_x, y + c_y) \mid x = \sum_{i=0}^{3}\sum_{j=0}^{3} D^u_{ij}(u,v)\delta_{ij},\, y = \sum_{i=0}^{3}\sum_{j=0}^{3} D^v_{ij}(u,v)\delta_{ij}\},$$

where $c = (c_x, c_y)$, $|\delta_{ij}| \leq \delta$, and $0 \leq u, v \leq 1$. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Then, $S_2(e_1,\delta_x)$ is the set of all possible values of r_1 from the configurations of control points ϕ_{ij} that satisfy $|\Delta x_{ij}| \leq \delta_x$. Similarly, $S_2(e_2,\delta_y)$ contains all possible values of r_2 under the constraint $|\Delta y_{ij}| \leq \delta_y$. Fig. 1 shows a schematic diagram of $S_2(e_1,\delta_x)$ and $S_2(e_2,\delta_y)$.

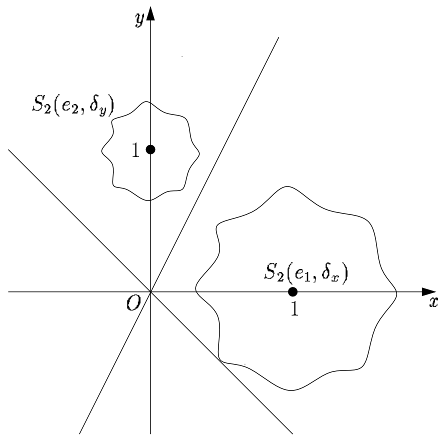


Figure 1. A schematic diagram of $S_2(e_1,\delta_x)$ and $S_2(e_2,\delta_y)$.

Let $\delta_x = \max\{|\Delta x_{ij}|\}$ and $\delta_y = \max\{|\Delta y_{ij}|\}$. Then, for any (u,v) in Ω_2, r_1 and r_2 are contained in $S_2(e_1,\delta_x)$ and $S_2(e_2,\delta_y)$, respectively. Suppose that no line simultaneously intersects the origin, $S_2(e_1,\delta_x)$, and $S_2(e_2,\delta_y)$

as shown in Fig. 1. In this case, for each (u, v) in Ω_2, no line can simultaneously passes through the origin, r_1, and r_2 and hence function f_2 is locally injective. The following lemma summarizes the relationship of $S_2(e_1, \delta_x)$ and $S_2(e_2, \delta_y)$ with the local injectivity of function f_2.

Lemma 1 *Function f_2 is locally injective all over the domain if no line simultaneously passes through the origin, $S_2(e_1, \delta_x)$, and $S_2(e_2, \delta_y)$.*

To determine the values of δ_x and δ_y that satisfy the condition in Lemma 1, it is necessary to represent the shape of $S_2(c, \delta)$ in terms of c and δ. Since it is not easy to analyze the exact shape of $S_2(c, \delta)$, we find two simple regions in R^2 that include $S_2(c, \delta)$. The shape of $S_2(c, \delta)$ can then be approximated by the intersection of the two regions.

3.2. Bounding regions of $S_2(c, \delta)$

Let $RS(c, \delta)$ be a rectangular region in R^2 defined by

$$RS(c,\delta) = \{(x+c_x, y+c_y) \,|\, |x| \le \frac{3}{2}\delta, |y| \le \frac{3}{2}\delta\},$$

where $c = (c_x, c_y)$. Let O denote the origin in R^2. Fig. 2(a) shows the configuration of $RS(O, \delta)$. It is simple to verify that $RS(c, \delta)$ is a bounding region of $S_2(c, \delta)$ by using the properties of uniform cubic B-spline basis functions. In the proofs of the following lemmas about bounding regions, without loss of generality, we only consider the case when c is the origin O.

Lemma 2 $S_2(c, \delta) \subset RS(c, \delta)$.

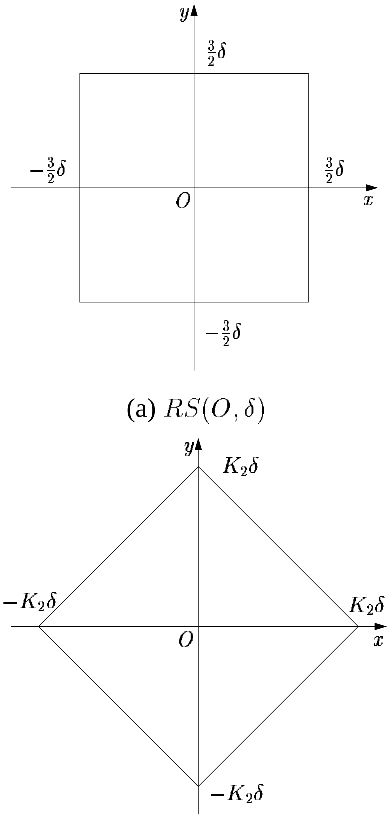
(Proof) For each point $(x, y) \in S_2(O, \delta)$, we have

$$\begin{aligned} |x| &= \left| \sum_{i=0}^{3}\sum_{j=0}^{3} D^u_{ij}(u,v)\delta_{ij} \right| \\ &\le \sum_{i=0}^{3} |B_i'(u)| \sum_{j=0}^{3} B_j(v)\delta \\ &\le \frac{3}{2}\delta. \end{aligned}$$

Similarly, we can show that $|y| \le \frac{3}{2}\delta$. □

To obtain the second bounding region of $S_2(c, \delta)$, we introduce a constant K_2 that is related with uniform cubic B-spline basis functions:

$$\begin{aligned} K_2 &= \max_{0\le u,v\le 1} \left\{ \sum_{i=0}^{3}\sum_{j=0}^{3} |D^u_{ij}(u,v) + D^v_{ij}(u,v)| \right\} \\ &= \max_{0\le u,v\le 1} \left\{ \sum_{i=0}^{3}\sum_{j=0}^{3} |D^u_{ij}(u,v) - D^v_{ij}(u,v)| \right\}. \end{aligned}$$



(a) $RS(O, \delta)$

(b) $RD(O, \delta)$

Figure 2. Regions $RS(O, \delta)$ and $RD(O, \delta)$.

Note that $B_j(v) = B_{3-j}(1-v)$ and $B_j'(v) = -B_{3-j}'(1-v)$. To compute the value of K_2, the domain $0 \le u, v \le 1$ is partitioned to a very dense grid with the grid spacing 10^{-10}. We then evaluate $\sum_{i=0}^{3}\sum_{j=0}^{3} |D^u_{ij}(u,v) + D^v_{ij}(u,v)|$ at every grid point, and the maximum occurs at (u_0, u_0), where $u_0 = 0.2448210078$ or $u_0 = 0.7551789922$. Then, K_2 is approximately 2.046392675.

Let $RD(c, \delta)$ be a region in R^2 defined by

$$\begin{aligned} &RD(c,\delta) \\ &\quad = \{(x+c_x, y+c_y) \,|\, |x+y| \le K_2\delta, |x-y| \le K_2\delta\}. \end{aligned}$$

Fig. 2(b) shows the configuration of $RD(O, \delta)$. The following lemma shows that $RD(c, \delta)$ is a bounding region of $S_2(c, \delta)$.

Lemma 3 $S_2(c, \delta) \subset RD(c, \delta)$.

(Proof) Let $R_z(S_2(O, \delta))$ be the region in R^2 that corresponds to the rotation of $S_2(O, \delta)$ by $\frac{\pi}{4}$ with respect to the origin.

$$\begin{aligned} &R_z(S_2(O,\delta)) \\ &\quad = \{(x', y') \,|\, x' = \frac{1}{\sqrt{2}}(x-y), y' = \frac{1}{\sqrt{2}}(x+y)\}, \end{aligned}$$

where $(x, y) \in S_2(O, \delta)$. Let $R_z(RD(O, \delta))$ be the rotation of $RD(O, \delta)$ by $\frac{\pi}{4}$ with respect to the origin.

$$R_z(RD(O, \delta)) \quad = \quad \{(x', y') | |x'| \leq \frac{K_2}{\sqrt{2}}\delta, |y'| \leq \frac{K_2}{\sqrt{2}}\delta\}.$$

For each point $(x', y') \in R_z(S_2(O, \delta))$, we have

$$\begin{aligned} |x'| &= \left| \frac{1}{\sqrt{2}} \sum_{i=0}^{3} \sum_{j=0}^{3} (D_{ij}^u(u, v) - D_{ij}^v(u, v))\delta_{ij} \right| \\ &\leq \frac{1}{\sqrt{2}} \sum_{i=0}^{3} \sum_{j=0}^{3} |D_{ij}^u(u, v) - D_{ij}^v(u, v)||\delta_{ij}| \\ &\leq \frac{K_2}{\sqrt{2}}\delta. \end{aligned}$$

Similarly, we can show that $|y'| \leq \frac{K_2}{\sqrt{2}}\delta$. Then we have $R_z(S_2(O, \delta)) \subset R_z(RD(O, \delta))$, which implies $S_2(O, \delta) \subset RD(O, \delta)$. □

Let $BP(c, \delta)$ be the intersection of $RS(c, \delta)$ and $RD(c, \delta)$:

$$BP(c, \delta) \quad = \quad RS(c, \delta) \cap RD(c, \delta).$$

Fig. 3(a) shows the configuration of $BP(O, \delta)$. It trivially holds that $BP(c, \delta)$ is a polygonal bounding region of $S_2(c, \delta)$.

Lemma 4 $S_2(c, \delta) \subset BP(c, \delta)$.

Let $BC(c, \delta)$ be a circular region in R^2 defined by

$$BC(c, \delta) \quad = \quad \{(x + c_x, y + c_y) | x^2 + y^2 \leq (A_2\delta)^2\},$$

where $c = (c_x, c_y)$ and

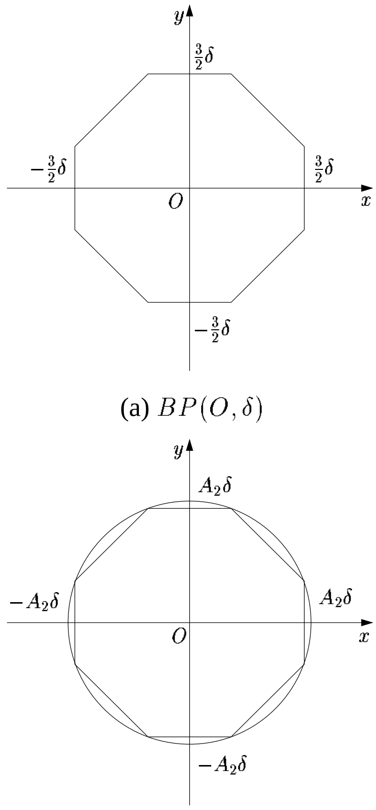
$$A_2 \quad = \quad \sqrt{(\frac{3}{2})^2 + (K_2 - \frac{3}{2})^2}.$$

$BC(c, \delta)$ is the smallest circular region that encloses $BP(c, \delta)$. Fig. 3(b) shows the configuration of $BC(O, \delta)$ overlaid on $BP(O, \delta)$. It is obvious that $BC(c, \delta)$ is a circular bounding region of $S_2(c, \delta)$.

Lemma 5 $S_2(c, \delta) \subset BC(c, \delta)$.

3.3. Sufficient conditions for 2D local injectivity

With the geometric interpretation presented in Section 3.1 and the bounding regions obtained in Section 3.2, we can derive sufficient conditions for the local injectivity of a 2D uniform cubic B-spline function. Let f_2 be the function defined by Eq. (1) and $\Delta\phi_{ij} = \phi_{ij} - \phi_{ij}^0 = (\Delta x_{ij}, \Delta y_{ij})$ for $i, j = 0, 1, 2, 3$. Let $\delta_x = \max\{|\Delta x_{ij}|\}$ and $\delta_y = \max\{|\Delta y_{ij}|\}$.



(b) $BC(O, \delta)$ overlaid on $BP(O, \delta)$

Figure 3. Regions $BP(O, \delta)$ and $BC(O, \delta)$.

Theorem 1 *Function f_2 is locally injective all over the domain if $\delta_x < \frac{1}{K_2}$ and $\delta_y < \frac{1}{K_2}$.*

(Proof) Suppose that $\delta_x < \frac{1}{K_2}$ and $\delta_y < \frac{1}{K_2}$. Then we have $BP(e_1, \delta_x) \subset \{(x, y) | x + y > 0, x - y > 0\}$ and $BP(e_2, \delta_y) \subset \{(x, y) | x + y > 0, x - y < 0\}$ (See Fig. 4). Hence, no line that passes through the origin can intersect both $BP(e_1, \delta_x)$ and $BP(e_2, \delta_y)$. From Lemma 4, this implies that no line simultaneously passes through the origin, $S_2(e_1, \delta_x)$ and $S_2(e_2, \delta_y)$. Then, from Lemma 1, function f_2 is locally injective all over the domain. □

In Theorem 1, the sufficient condition for local injectivity provides the same bound for the x- and y-displacements of control points. This condition cannot cover the case in which function f_2 may be locally injective if δ_y is much smaller than $\frac{1}{K_2}$ though $\delta_x \geq \frac{1}{K_2}$. We can obtain a sufficient condition that can handle this case by using the circular bounding region $BC(c, \delta)$. We first show a lemma related with a line passing through the origin and regions $BC(e_1, \delta_x)$ and $BC(e_2, \delta_y)$.

Lemma 6 *If there is a line that simultaneously passes*

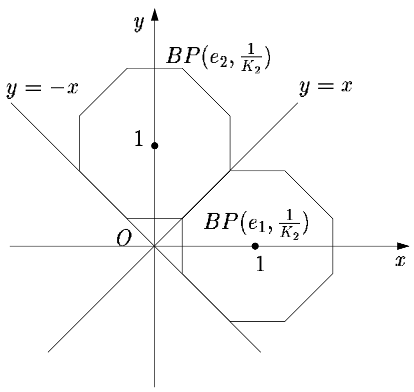


Figure 4. The configuration of $BP(e_1, \frac{1}{K_2})$ and $BP(e_2, \frac{1}{K_2})$.

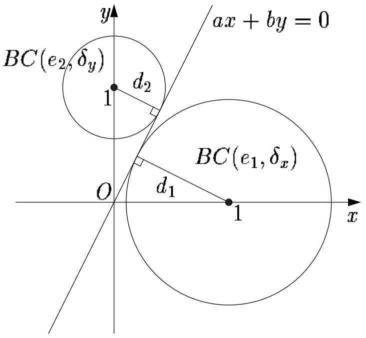


Figure 5. A configuration of $BC(e_1, \delta_x)$ and $BC(e_2, \delta_y)$ with a line passing through the origin.

through the origin, $BC(e_1, \delta_x)$, and $BC(e_2, \delta_y)$, then $\delta_x^2 + \delta_y^2 \geq (\frac{1}{A_2})^2$.

(Proof) Let l be a line that passes through the origin, which is represented by $ax + by = 0$. Let d_1 and d_2 be the distances of e_1 and e_2 from line l, respectively. Then we have $d_1 = \frac{|a|}{\sqrt{a^2+b^2}}$ and $d_2 = \frac{|b|}{\sqrt{a^2+b^2}}$, which gives $d_1^2 + d_2^2 = 1$. Suppose that line l simultaneously intersects $BC(e_1, \delta_x)$ and $BC(e_2, \delta_y)$. Fig. 5 shows an example of such a configuration. Then we have $d_1 \leq A_2\delta_x$ and $d_2 \leq A_2\delta_y$. This implies that $1 = d_1^2 + d_2^2 \leq A_2^2(\delta_x^2 + \delta_y^2)$. □

Theorem 2 *Function f_2 is locally injective all over the domain if $\delta_x^2 + \delta_y^2 < (\frac{1}{A_2})^2$.*

(Proof) Suppose that $\delta_x^2 + \delta_y^2 < (\frac{1}{A_2})^2$. Then, from Lemma 6, there cannot be a line passing through the origin that simultaneously intersects $BC(e_1, \delta_x)$ and $BC(e_2, \delta_y)$. This implies from Lemmas 1 and 5 that function f_2 is locally injective all over the domain. □

Since constant K_2 is approximately 2.046392675, the bounds $\frac{1}{K_2}$ and $(\frac{1}{A_2})^2$ given in Theorems 1 and 2 are approximately 0.488664767 and 0.392380757, respectively. Then it seems that the sufficient conditions in Theorems 1 and 2 only provide very small bounds for control point displacements. However, by using the affine invariance property of B-splines, we can extend Theorems 1 and 2 to handle general cases with possibly large control point displacements. That is, the bounds can be applied to the normalized values of control point displacements instead of the given values.

To normalize the control point displacements, we consider a 2D affine transformation M_2 that moves the given control points ϕ_{ij} toward the canonical positions ϕ_{ij}^0. Let ϕ'_{ij} be the new position of a control point ϕ_{ij} when it is transformed by M_2. We determine the affine transformation M_2 so that it minimizes the approximation error, $\sum_{i=0}^{3}\sum_{j=0}^{3} ||\phi'_{ij} - \phi_{ij}^0||^2$. Such a transformation M_2 can be obtained by simple linear algebra [11]. Let $\Delta\phi'_{ij} = \phi'_{ij} - \phi_{ij}^0 = (\Delta x'_{ij}, \Delta y'_{ij})$. Let $\delta'_x = \max\{|\Delta x'_{ij}|\}$ and $\delta'_y = \max\{|\Delta y'_{ij}|\}$.

Theorem 3 *Function f_2 is locally injective all over the domain if the 2D affine transformation M_2 is invertible and if δ'_x and δ'_y satisfy one of the conditions of $\delta'_x < \frac{1}{K_2}$ and $\delta'_y < \frac{1}{K_2}$ or $(\delta'_x)^2 + (\delta'_y)^2 < (\frac{1}{A_2})^2$.*

(Proof) Let f'_2 be the 2D uniform cubic B-spline function determined by transformed control points ϕ'_{ij}. If $\delta'_x < \frac{1}{K_2}$ and $\delta'_y < \frac{1}{K_2}$ or if $(\delta'_x)^2 + (\delta'_y)^2 < (\frac{1}{A_2})^2$, function f'_2 is locally injective all over the domain from Theorem 1 or Theorem 2, respectively. Due to the affine invariance property of B-splines, function value $f_2(u, v)$ is mapped to function value $f'_2(u, v)$ by transformation M_2. If M_2 is invertible, the mapping between $f_2(u, v)$ and $f'_2(u, v)$ is a one-to-one correspondence. □

For example, when control point displacements $\Delta\phi_{ij}$ are $(5, 5)$ for all i, j, it is obvious that function f_2 is locally injective all over the domain. However, Theorems 1 and 2 do not cover this case because the conditions in them are not satisfied with $\Delta\phi_{ij} = (5, 5)$. In contrast, when we normalize the displacements $\Delta\phi_{ij}$, transformation M_2 is a translation by $(-5, -5)$ and $\Delta\phi'_{ij} = (0, 0)$ for all i, j, which obviously satisfies the condition in Theorem 3.

The sufficient conditions in Theorems 1 and 2 are not necessary for the local injectivity of function f_2 even if we apply them to the normalized control point displacements. For example, let $\Delta x_{0j} = \Delta x_{1j} = -\Delta x_{2j} = -\Delta x_{3j} = 0.65$ and $\Delta y_{ij} = 0$, for $i, j = 0, 1, 2, 3$. In this case, both of the conditions in Theorems 1 and 2 are violated but function f_2 is still locally injective.

However, the bound for the control point displacements in Theorem 1 is tight. Lee *et al*. [9] presented a control lattice configuration with which function f_2 is not locally injective when $\delta_x = \delta_y = \frac{1}{K_2}$. We have not fully investigated the tightness of the inequality in Theorem 2 yet.

Theorem 2 is more useful than Theorem 1 when δ_x and δ_y are different. For example, given a control lattice with $\delta_x = 0.62$ and $\delta_y = 0$, we cannot verify the local injectivity of the resulting function f_2 with Theorem 1. In contrast, Theorem 2 can be used to guarantee that the function f_2 is locally injective.

4. Local Injectivity Conditions of 3D B-spline Functions

In this section, we present sufficient conditions for the local injectivity of a 3D uniform cubic B-spline function by expanding the geometric interpretation and the bounding regions presented in Section 3 to 3D space.

4.1. Geometric interpretation of 3D local injectivity

Let f_3 be a 3D uniform cubic B-spline function defined by Eq. (2). When ϕ_{ijk} equals $\phi^0_{ijk} = (i-1, j-1, k-1)$ for $i, j, k = 0, 1, 2, 3$, function f_3 is reduced to an identity function. Let $\Delta\phi_{ijk}$ be the displacement of control point ϕ_{ijk} from ϕ^0_{ijk}, that is, $\Delta\phi_{ijk} = \phi_{ijk} - \phi^0_{ijk} = (\Delta x_{ijk}, \Delta y_{ijk}, \Delta z_{ijk})$. Then we have

$$f_3(u,v,w) = (u,v,w) + \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} B_i(u)B_j(v)B_k(w)\Delta\phi_{ijk}.$$

Let $r_1 = (\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial x}{\partial w})$, $r_2 = (\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial y}{\partial w})$, and $r_3 = (\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w})$ denote the row vectors of Jacobian matrix $J(f_3)$. Let $D^u_{ijk}(u,v,w) = B'_i(u)B_j(v)B_k(w)$, $D^v_{ijk}(u,v,w) = B_i(u)B'_j(v)B_k(w)$, and $D^w_{ijk}(u,v,w) = B_i(u)B_j(v)B'_k(w)$. We define $S_3(c,\delta)$ as a region in R^3 such that

$$S_3(c,\delta) = \{(x+c_x, y+c_y, z+c_z) \mid \\ x = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} D^u_{ijk}(u,v,w)\delta_{ijk}, \\ y = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} D^v_{ijk}(u,v,w)\delta_{ijk}, \\ z = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} D^w_{ijk}(u,v,w)\delta_{ijk}\},$$

where $c = (c_x, c_y, c_z)$, $|\delta_{ijk}| \le \delta$, and $0 \le u, v, w \le 1$. $S_3(c,\delta)$ is the 3D expansion of the region $S_2(c,\delta)$. Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$. Fig. 6 shows a schematic diagram of the regions $S_3(e_1,\delta_x)$, $S_3(e_2,\delta_y)$, and $S_3(e_3,\delta_z)$.

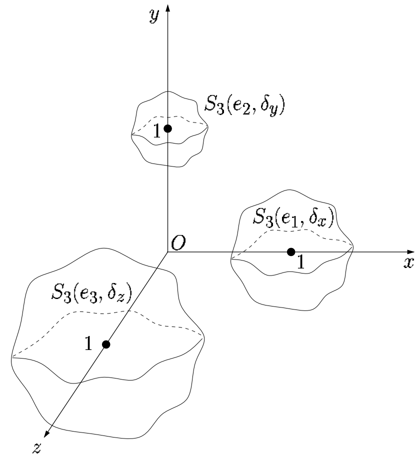


Figure 6. A schematic diagram of $S_3(e_1,\delta_x)$, $S_3(e_2,\delta_y)$, and $S_3(e_3,\delta_z)$.

Let $\delta_x = \max\{|\Delta x_{ijk}|\}$, $\delta_y = \max\{|\Delta y_{ijk}|\}$, and $\delta_z = \max\{|\Delta z_{ijk}|\}$. The 3D extension of Lemma 1 can easily be proved from the fact that three vectors r_1, r_2, and r_3 in R^3 are linearly independent if and only if there is no plane passing through the origin that contains the points r_1, r_2, and r_3. As in the 2D case, we interchangeably use a 3D vector and a point in R^3.

Lemma 7 *Function f_3 is locally injective all over the domain if no plane simultaneously passes through the origin, $S_3(e_1,\delta_x)$, $S_3(e_2,\delta_y)$, and $S_3(e_3,\delta_z)$.*

4.2. Bounding regions of $S_3(c,\delta)$

Let $RTS(c,\delta)$ be a region in R^3 defined by

$$RTS(c,\delta) = \{(x+c_x, y+c_y, z+c_z) \mid \\ |x| \le \frac{3}{2}\delta, |y| \le \frac{3}{2}\delta, |z| \le \frac{3}{2}\delta\},$$

where $c = (c_x, c_y, c_z)$. It can be proved in a similar way to Lemma 2 that $RTS(c,\delta)$ is a bounding region of $S_3(c,\delta)$.

Lemma 8 $S_3(c,\delta) \subset RTS(c,\delta)$.

Let $RTD(c,\delta)$ be a region in R^3 defined by

$$RTD(c,\delta) = RTX(c,\delta) \cap RTY(c,\delta) \cap RTZ(c,\delta),$$

where

$$RTX(c,\delta) = \{(x+c_x, y+c_y, z+c_z) | \\ |x| \le \frac{3}{2}\delta, |y+z| \le K_2\delta, |y-z| \le K_2\delta\},$$
$$RTY(c,\delta) = \{(x+c_x, y+c_y, z+c_z) | \\ |y| \le \frac{3}{2}\delta, |z+x| \le K_2\delta, |z-x| \le K_2\delta\},$$
$$RTZ(c,\delta) = \{(x+c_x, y+c_y, z+c_z) | \\ |z| \le \frac{3}{2}\delta, |x+y| \le K_2\delta, |x-y| \le K_2\delta\},$$

and $c = (c_x, c_y, c_z)$. Note that $RTZ(c,\delta)$ is an extrusion of the 2D region $RD(c,\delta)$ in the z-direction. Similarly, $RTX(c,\delta)$ and $RTY(c,\delta)$ are extrusions of $RD(c,\delta)$ when it is defined in the yz- and zx-planes, respectively. It is simple to show that $RTD(c,\delta)$ is a bounding region of $S_3(c,\delta)$.

Lemma 9 $S_3(c,\delta) \subset RTD(c,\delta)$.

(Proof) It can be shown that $S_3(O,\delta) \subset RTZ(O,\delta)$ in a similar way to Lemma 3. Also, we can show that $S_3(O,\delta) \subset RTX(O,\delta)$ and $S_3(O,\delta) \subset RTY(O,\delta)$ by using rotations with respect to the x- and y-axes, respectively. □

To obtain a regular octahedron that includes $S_3(c,\delta)$, we introduce a constant K_3, which is the 3D correspondence of K_2:

$$K_3 = \max_{0\le u,v,w\le 1} \left\{ \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} |G_{ijk}(u,v,w)| \right\},$$

where

$$G_{ijk}(u,v,w) \\ = D^u_{ijk}(u,v,w) + D^v_{ijk}(u,v,w) + D^w_{ijk}(u,v,w).$$

To compute the value of K_3, the domain $0 \le u,v,w \le 1$ is partitioned to a very dense grid with the grid spacing 10^{-10}. We then evaluate $\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} |G_{ijk}(u,v,w)|$ at every grid point, and the maximum occurs at (u_0, u_0, u_0), where $u_0 = 0.1640662347$ or $u_0 = 0.8359337653$. Then, K_3 is approximately 2.479472335.

Let $RTO(c,\delta)$ be a region in R^3 defined by

$$RTO(c,\delta) = \{(x+c_x, y+c_y, z+c_z) | \\ |x+y+z| \le K_3\delta, |x+y-z| \le K_3\delta, \\ |x-y+z| \le K_3\delta, |x-y-z| \le K_3\delta\},$$

where $c = (c_x, c_y, c_z)$. The following lemma shows that $RTO(c,\delta)$ is a bounding region of $S_3(c,\delta)$.

Lemma 10 $S_3(c,\delta) \subset RTO(c,\delta)$.

(Proof) We can represent $RTO(O,\delta)$ by

$$RTO(O,\delta) \\ = RTP_1(O,\delta) \cap RTP_2(O,\delta) \cap RTP_3(O,\delta) \cap RTP_4(O,\delta),$$

where

$$RTP_1(O,\delta) = \{(x,y,z) || x+y+z| \le K_3\delta\},$$
$$RTP_2(O,\delta) = \{(x,y,z) || x+y-z| \le K_3\delta\},$$
$$RTP_3(O,\delta) = \{(x,y,z) || x-y+z| \le K_3\delta\},$$
$$RTP_4(O,\delta) = \{(x,y,z) || x-y-z| \le K_3\delta\}.$$

Consider a 3D rotation R_{zx} that maps the vector $(1,1,1)$ onto the z-axis. Rotation R_{zx} transforms $RTP_1(O,\delta)$ to $R_{zx}(RTP_1(O,\delta))$, where

$$R_{zx}(RTP_1(O,\delta)) = \{(x',y',z') || z'| \le \frac{K_3}{\sqrt{3}}\delta\}.$$

Let z' be the z-coordinate of the new position when we apply rotation R_{zx} to a point $(x,y,z) \in S_3(O,\delta)$. Then we have

$$|z'| = \left|\frac{1}{\sqrt{3}}(x+y+z)\right| \\ \le \frac{1}{\sqrt{3}} \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} |G_{ijk}(u,v,w)||\delta_{ijk}| \\ \le \frac{K_3}{\sqrt{3}}\delta,$$

which implies $S_3(O,\delta) \subset RTP_1(O,\delta)$. Similarly, we can show that $S_3(O,\delta) \subset RTP_2(O,\delta)$, $S_3(O,\delta) \subset RTP_3(O,\delta)$, and $S_3(O,\delta) \subset RTP_4(O,\delta)$. □

Let $BPH(c,\delta)$ be the intersection of $RTS(c,\delta)$, $RTD(c,\delta)$, and $RTO(c,\delta)$:

$$BPH(c,\delta) = RTS(c,\delta) \cap RTD(c,\delta) \cap RTO(c,\delta).$$

Fig. 7 shows the shape of $BPH(O,1)$ from several different viewpoints. The boundary of $BPH(c,\delta)$ consists of 26 faces. It trivially holds that $BPH(c,\delta)$ is a polyhedral bounding region of $S_3(c,\delta)$.

Lemma 11 $S_3(c,\delta) \subset BPH(c,\delta)$.

Let $BSP(c,\delta)$ be a spherical region defined by

$$BSP(c,\delta) \\ = \{(x+c_x, y+c_y, z+c_z) | x^2+y^2+z^2 \le (A_3\delta)^2\},$$

where $c = (c_x, c_y, c_z)$ and

$$A_3 = \sqrt{(\frac{3}{2})^2 + (K_2 - \frac{3}{2})^2 + (K_3 - K_2)^2}.$$

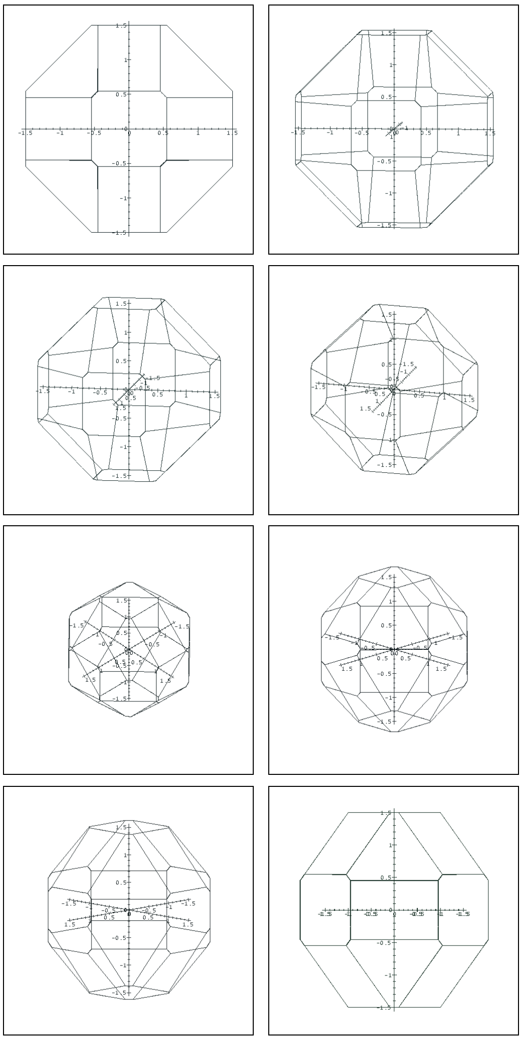


Figure 7. Region $BPH(O, 1)$ from different viewpoints.

$BSP(c, \delta)$ is the smallest spherical region that encloses $BPH(c, \delta)$. It is obvious that $BSP(c, \delta)$ is a bounding region of $S_3(c, \delta)$.

Lemma 12 $S_3(c, \delta) \subset BSP(c, \delta)$.

4.3. Sufficient conditions for 3D local injectivity

Now we present sufficient conditions for the local injectivity of a 3D uniform cubic B-spline function. Let f_3 be the function defined by Eq. (2) and $\Delta\phi_{ijk} = \phi_{ijk} - \phi^0_{ijk} = (\Delta x_{ijk}, \Delta y_{ijk}, \Delta z_{ijk})$ for $i, j, k = 0, 1, 2, 3$. Let $\delta_x = \max\{|\Delta x_{ijk}|\}$, $\delta_y = \max\{|\Delta y_{ijk}|\}$, and $\delta_z = \max\{|\Delta z_{ijk}|\}$.

Theorem 4 *Function f_3 is locally injective all over the domain if $\delta_x < \frac{1}{K_3}$, $\delta_y < \frac{1}{K_3}$, and $\delta_z < \frac{1}{K_3}$.*

(Proof) By using the polyhedral bounding region $BPH(c, \delta)$ of $S_3(c, \delta)$, this theorem can be proved in a similar way to Theorem 1. □

Theorem 5 *Function f_3 is locally injective all over the domain if $\delta_x^2 + \delta_y^2 + \delta_z^2 < (\frac{1}{A_3})^2$.*

(Proof) By using the spherical bounding region $BSP(c, \delta)$ of $S_3(c, \delta)$, this theorem can be proved in a similar way to Theorem 2. □

To normalize the control point displacements, we consider a 3D affine transformation M_3. Let ϕ'_{ijk} be the new position of a control point ϕ_{ijk} when it is transformed by M_3. We determine the affine transformation M_3 so that it minimizes the approximation error, $\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} ||\phi'_{ijk} - \phi^0_{ijk}||^2$. Let $\Delta\phi'_{ijk} = \phi'_{ijk} - \phi^0_{ijk} = (\Delta x'_{ijk}, \Delta y'_{ijk}, \Delta z'_{ijk})$. Let $\delta'_x = \max\{|\Delta x'_{ijk}|\}$, $\delta'_y = \max\{|\Delta y'_{ijk}|\}$, and $\delta'_z = \max\{|\Delta z'_{ijk}|\}$.

Theorem 6 *Function f_3 is locally injective all over the domain if the 3D affine transformation M_3 is invertible and if δ'_x, δ'_y, and δ'_z satisfy one of the conditions of $\delta'_x < \frac{1}{K_3}$, $\delta'_y < \frac{1}{K_3}$, and $\delta'_z < \frac{1}{K_3}$ or $(\delta'_x)^2 + (\delta'_y)^2 + (\delta'_z)^2 < (\frac{1}{A_3})^2$.*

(Proof) By using the affine invariance property of B-splines, this theorem can be proved in a similar way to Theorem 3. □

As in the 2D case, the sufficient conditions in Theorems 4 and 5 are not necessary for the local injectivity of function f_3. However, similar to Theorem 1, Theorem 4 provides a tight bound for the control point displacements. In Section 3.3, we mentioned there is a control lattice configuration with which function f_2 is not locally injective when $\delta_x = \delta_y = \frac{1}{K_2}$. By expanding the control lattice configuration to 3D, we can obtain a control lattice configuration with which f_3 is not locally injective when $\delta_x = \delta_y = \delta_z = \frac{1}{K_3}$. We have not fully investigated the tightness of the inequality in Theorem 5 yet.

5. Conclusions

In this paper, we have presented sufficient conditions for the local injectivity of 2D and 3D uniform cubic B-spline functions. We first proposed a geometric interpretation of the local injectivity of a uniform cubic B-spline function. Row vectors in the Jacobian matrix are mapped to regions in the space, for which we obtained simple bounding regions. Sufficient conditions represented in terms of control point displacements were finally derived by using the bounding regions. These sufficient conditions are simple and easy to check and will be useful to guarantee the injectivity of mapping functions in applications such as image warping and morphing, 3D deformation, and volume morphing.

Future work will include an investigation of sufficient conditions that consider each control point displacement independently. In this paper, sufficient conditions are represented only in terms of the maximums of the control point displacements in principal directions. The ultimate future goal will be to obtain simple and easy-to-check necessary and sufficient conditions for the injectivity of uniform cubic B-spline functions.

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