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Injectivity Conditions of 2D and 3D Uniform Cubic B-Spline Functions

Yongchoel Choi and Seungyong Lee[1]

Department of Computer Science and Engineering, Pohang University of Science and Technology (POSTECH), Pohang, 790-784, Korea

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Uniform cubic B-spline functions have been used for mapping functions in various areas such as image warping and morphing, 3D deformation, and volume morphing. The injectivity (one-to-one property) of a mapping function is crucial to obtaining desirable results in these areas. This paper considers the injectivity conditions of 2D and 3D uniform cubic B-spline functions. We propose a geometric interpretation of the injectivity of a uniform cubic B-spline function, with which 2D and 3D cases can be handled in a similar way. Based on our geometric interpretation, we present sufficient conditions for injectivity which are represented in terms of control point displacements. These sufficient conditions can be easily tested and will be useful in guaranteeing the injectivity of mapping functions in application areas.

1. INTRODUCTION

In computer graphics, mapping functions that transform certain domains into themselves are widely used. In image warping and morphing, an image is distorted by a 2D mapping function that provides a new position for each point in the image [14]. In deformation techniques such as free-form deformations [12], 3D mapping functions are used to determine the deformed positions of object points. In volume morphing, user-specified features are aligned by distorting given volumes with 3D mapping functions [2, 10].

In these areas, the injectivity (one-to-one property) of a mapping function is essential to obtaining good results. In image warping and morphing, if a mapping function is not injective, the resulting distorted image may contain undesirable wrinkles because parts of the original image are folded upon nearby parts. Several techniques have been developed to generate injective mapping functions for image warping and morphing [3, 7–9].

[1] To whom correspondence should be addressed. E-mail: leesy@postech.ac.kr. URL:http://www.postech.ac.kr/~leesy. Fax: +82-54-279-5699.

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In deformation techniques, the injectivity of a mapping function guarantees that no self-intersection is introduced to an object in the deformation process. In volume morphing, there is no ambiguity in determining the voxel values of a distorted volume provided that the mapping function is injective.

Due to their local control property and simplicity, uniform cubic B-spline functions have been used for mapping functions in image morphing and 3D deformation. A 2D uniform cubic B-spline function is defined by applying uniform cubic B-spline bases to 2D control points in a 2D control lattice. Similarly, a 3D uniform cubic B-spline function is determined by a 3D parallelepiped control lattice that consists of 3D control points. Note that each of the 2D and 3D B-spline functions is different from a B-spline surface, which is a function from 2D to 3D and is obtained from a 2D control lattice with 3D control points. Lee *et al.* used 2D B-spline functions to efficiently generate mapping functions in image morphing [8, 9]. Three-dimensional B-spline functions have been adopted to develop direct manipulation techniques for free-form deformations [5, 6].

Figure 1 gives an example of the application of 2D B-spline functions to image warping. Figure 1a is the original image with a control lattice overlaid on it. In Fig. 1b, the control lattice is changed to obtain a warped image. Figure 1c shows that a noninjective B-spline function generates an undesirable foldover, where the nose is covered with the mouth. In Fig. 2, 3D B-spline functions are used to deform a 3D object. Figure 2a shows a shark with a control lattice overlaid around the tail. The tail can be deformed by manipulating the control lattice, as shown in Fig. 2b. In Fig. 2c, the 3D B-spline function resulting from the changed control lattice is not injective and the back part of the tail penetrates the front part in the deformed shape.

To obtain injective mapping functions in image morphing, Lee *et al.* presented a sufficient condition for the injectivity of a 2D uniform cubic B-spline function [8, 9]. The sufficient condition provides a single bound for the displacements of control points that guarantees the injectivity of a 2D B-spline function. However, the condition cannot cover the cases in which several control point displacements are above the bound and other are far below the bound while the resulting function is still injective. Goodman and Unsworth proposed a sufficient condition for a 2D Bézier surface to be injective [4], which can be applied to a 2D B-spline function. For an $m \times n$ lattice of control points, the condition contains $2m(m+1) + 2n(n+1)$ linear inequalities. Unfortunately, when the number of control points is large, the time to check the condition becomes prohibitive. Although it could be useful in 3D deformation and volume morphing, there has been no research on an injectivity condition for a 3D B-spline function.

In this paper, we consider the injectivity conditions of 2D and 3D uniform cubic B-spline functions. We first propose a geometric interpretation of the injectivity of a 2D B-spline function. Based on this geometric interpretation, we obtain novel sufficient conditions for the injectivity of a 2D B-spline function. The sufficient conditions are represented by inequalities of control point displacements and cover more cases than the previous result [8, 9]. To examine the injectivity condition of a 3D B-spline function, which has not been explored yet, we expand the geometric interpretation of injectivity to 3D. Sufficient conditions for the injectivity of a 3D B-spline function are then obtained, which are also represented by inequalities of control point displacements.

The remainder of this paper is organized as follows. In Section 2, we review the mathematical preliminaries. Sections 3 and 4 consider the injectivity conditions of 2D and 3D B-spline functions, respectively. Section 5 concludes this paper with future work descriptions.

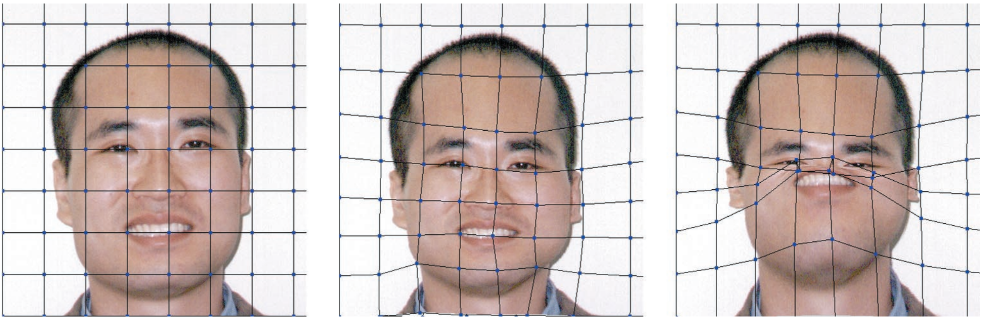

FIG. 1. 2D B-spline functions for image warping.

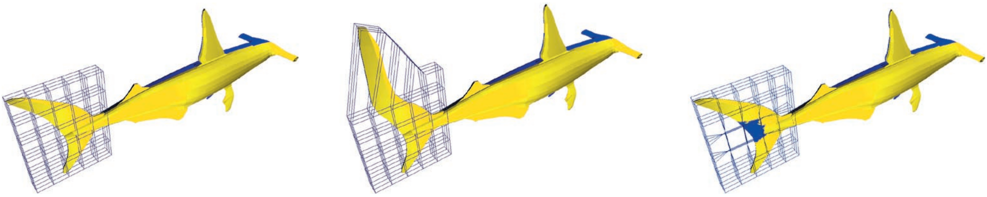

FIG. 2. 3D B-spline functions for deformation.

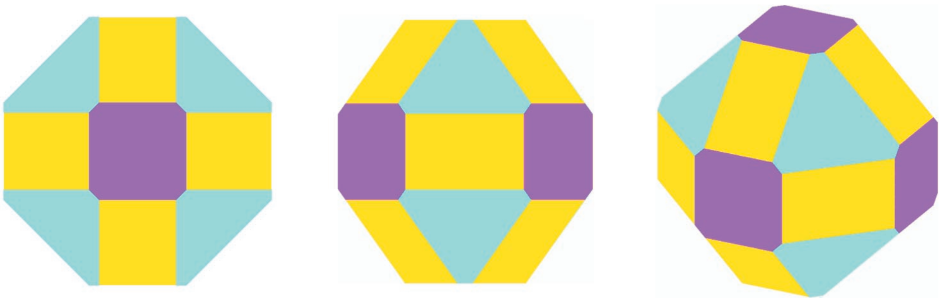

FIG. 9. Region $BPH(\mathbf{O}, 1)$ from different viewpoints.

2. MATHEMATICAL PRELIMINARIES

Let F_2 be a 2D uniform cubic B-spline function defined with an $(m+3) \times (n+3)$ control lattice Φ. Function F_2 consists of $m \times n$ 2D patches, each of which is determined by 4×4 control points in R^2. The injectivity of function F_2 may be violated in two cases. First, a global violation of the injectivity happens when a patch of F_2 intersects another patch. Due to the convex hull property of B-splines, this global violation is possible only if the control lattice Φ contains a self-intersection. Second, the injectivity may be violated locally in a patch even though the control lattice Φ is not self-intersecting. Although it is counterintuitive that a B-spline function may not be injective when a control lattice does not self-intersect, Lee *et al.* showed an example of such a configuration [9].

In applications of B-spline functions such as morphing and deformation, the global violation of injectivity is not allowed in most cases and can be prevented by using control lattices which are free from self-intersection. Hence, in this paper, we focus on the local injectivity of a B-spline function, which cannot be determined by checking the self-intersection of a control lattice.

When we investigate sufficient conditions for the local injectivity of function F_2, it is sufficient to consider only one patch of F_2 because the same conditions can be applied to all other patches. Without loss of generality, we represent a 2D uniform cubic B-spline function by a patch f_2, which is defined by

$$f_2(u, v) = (x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} B_i(u) B_j(v)\, \phi_{ij}, \tag{1}$$

where $0 \le u, v \le 1$. Uniform cubic B-spline basis functions, B_0, B_1, B_2, and B_3, are defined by

$$\begin{aligned} B_0(u) &= \frac{(1-u)^3}{6}, \\ B_1(u) &= \frac{3u^3 - 6u^2 + 4}{6}, \\ B_2(u) &= \frac{-3u^3 + 3u^2 + 3u + 1}{6}, \\ B_3(u) &= \frac{u^3}{6}, \end{aligned}$$

where $0 \le u \le 1$. $\phi_{ij} = (x_{ij}, y_{ij})$, $i, j = 0, 1, 2, 3$, are 4×4 control points that determine function f_2. Similarly, we represent a 3D uniform cubic B-spline function by a patch f_3, which is defined by

$$f_3(u, v, w) = (x, y, z) = \sum_{i=0}^{3} \sum_{j=0}^{3} \sum_{k=0}^{3} B_i(u) B_j(v) B_k(w)\, \phi_{ijk}, \tag{2}$$

where $0 \le u, v, w \le 1$.

Functions f_2 and f_3 are locally injective if and only if their Jacobian matrices are nonsingular all over the domain [1]. The Jacobian matrix of f_2 is defined by

$$J(f_2) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}.$$

The Jacobian matrix of function f_3 is defined by

$$J(f_3) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

It is known that a square matrix is nonsingular if and only if its row vectors are linearly independent [13]. Hence, function f_2 is locally injective if and only if two 2D vectors, $(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v})$ and $(\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v})$, are linearly independent all over the domain. Similarly, function f_3 is locally injective if and only if three 3D vectors, $(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial x}{\partial w})$, $(\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial y}{\partial w})$, and $(\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w})$, are linearly independent. In Sections 3 and 4, these conditions are interpreted geometrically to obtain sufficient conditions for the local injectivity of functions f_2 and f_3, respectively.

The following properties of uniform cubic B-spline basis functions and their derivatives are used in later sections.

- $B_i(u) \geq 0$, for $i = 0, 1, 2, 3$
- $\sum_{i=0}^{3} B_i(u) = 1$
- $B_i(u) = B_{3-i}(1-u)$, for $i = 0, 1, 2, 3$
- $\sum_{i=0}^{3} |B_i'(u)| = -2u^2 + 2u + 1 \leq \frac{3}{2}$
- $B_i'(u) = -B_{3-i}'(1-u)$, for $i = 0, 1, 2, 3$.

3. INJECTIVITY CONDITIONS OF 2D B-SPLINE FUNCTIONS

In this section, we first propose a geometric interpretation of the local injectivity condition of a 2D uniform cubic B-spline function f_2. The two row vectors in the Jacobian matrix $J(f_2)$ are mapped to two regions in R^2 such that f_2 is locally injective if no line simultaneously passes through the origin and the two regions. We then obtain sufficient conditions for the local injectivity of f_2 by computing control point displacements which guarantee such a configuration of the two regions.

3.1. *Geometric Interpretation of 2D Injectivity*

Let f_2 be a 2D uniform cubic B-spline function defined by Eq. (1). The injectivity of function f_2 is determined by the configuration of the 4×4 control points $\phi_{ij}, i, j = 0, 1, 2, 3$. When ϕ_{ij} equals $\phi_{ij}^0 = (i-1, j-1)$ for all i, j, function f_2 is reduced to an identity function. Let $\Delta\phi_{ij}$ be the displacement of control point ϕ_{ij} from ϕ_{ij}^0; that is $\Delta\phi_{ij} = \phi_{ij} - \phi_{ij}^0 = (\Delta x_{ij}, \Delta y_{ij})$. Then function f_2 can be represented by

$$f_2(u, v) = (u, v) + \sum_{i=0}^{3}\sum_{j=0}^{3} B_i(u)B_j(v)\Delta\phi_{ij}.$$

Now we have

$$\frac{\partial x}{\partial u} = 1 + \sum_{i=0}^{3}\sum_{j=0}^{3} D_{ij}^u(u, v)\Delta x_{ij},$$

$$\frac{\partial x}{\partial v} = \sum_{i=0}^{3}\sum_{j=0}^{3} D_{ij}^v(u, v)\Delta x_{ij},$$

$$\frac{\partial y}{\partial u} = \sum_{i=0}^{3}\sum_{j=0}^{3} D_{ij}^{u}(u,v)\Delta y_{ij},$$

$$\frac{\partial y}{\partial v} = 1 + \sum_{i=0}^{3}\sum_{j=0}^{3} D_{ij}^{v}(u,v)\Delta y_{ij},$$

where $D_{ij}^{u}(u,v) = B_i'(u)B_j(v)$ and $D_{ij}^{v}(u,v) = B_i(u)B_j'(v)$.

Let Ω_2 be a rectangular domain in R^2 that contains points (u, v) such that $0 \le u, v \le 1$. Let $\mathbf{r}_1 = (\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v})$ and $\mathbf{r}_2 = (\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v})$ denote the row vectors of the Jacobian matrix $J(f_2)$. Note that $\mathbf{r}_1$ and $\mathbf{r}_2$ are functions from Ω_2 to R^2 that depend on u and v. Function f_2 is locally injective all over the domain Ω_2 if and only if vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are linearly independent for each (u, v) in Ω_2.

A 2D vector (x, y) can be interpreted as a point (x, y) in R^2. In this paper, we use a 2D vector and a point in R^2 interchangeably. Two different 2D vectors (x_1, y_1) and (x_2, y_2) are linearly independent if and only if the line passing through the two points (x_1, y_1) and (x_2, y_2) does not intersect the origin. Hence, function f_2 is locally injective over Ω_2 if and only if no line simultaneously passes through the origin, $\mathbf{r}_1$, and $\mathbf{r}_2$ for any (u, v) in Ω_2.

Let $S_2(\mathbf{c}, \delta)$ be a region in R^2 defined by

$$S_2(\mathbf{c}, \delta) = \left\{ (x + c_x, y + c_y) \mid x = \sum_{i=0}^{3}\sum_{j=0}^{3} D_{ij}^{u}(u,v)\delta_{ij},\, y = \sum_{i=0}^{3}\sum_{j=0}^{3} D_{ij}^{v}(u,v)\,\delta_{ij} \right\},$$

where $\mathbf{c} = (c_x, c_y)$, $|\delta_{ij}| \le \delta$, and $0 \le u, v \le 1$. Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Then, $S_2(\mathbf{e}_1, \delta_x)$ is the set of all possible values of $\mathbf{r}_1$ from the configurations of control points ϕ_{ij} that satisfy $|\Delta x_{ij}| \le \delta_x$. Similarly, $S_2(\mathbf{e}_2, \delta_y)$ contains all possible values of $\mathbf{r}_2$ under the constraint $|\Delta y_{ij}| \le \delta_y$. Figure 3 shows a schematic diagram of $S_2(\mathbf{e}_1, \delta_x)$ and $S_2(\mathbf{e}_2, \delta_y)$.

Let $\delta_x = \max\{|\Delta x_{ij}|\}$ and $\delta_y = \max\{|\Delta y_{ij}|\}$. Then, for any (u, v) in Ω_2, $\mathbf{r}_1$ and $\mathbf{r}_2$ are contained in $S_2(\mathbf{e}_1, \delta_x)$ and $S_2(\mathbf{e}_2, \delta_y)$, respectively. Suppose that no line simultaneously intersects the origin, $S_2(\mathbf{e}_1, \delta_x)$, and $S_2(\mathbf{e}_2, \delta_y)$ as shown in Fig. 3. In this case, for each (u, v) in Ω_2, no line can simultaneously passes through the origin, $\mathbf{r}_1$, and $\mathbf{r}_2$ and hence function

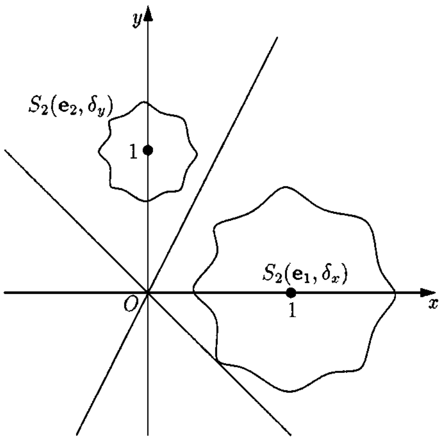

FIG. 3. A schematic diagram of $S_2(\mathbf{e}_1, \delta_x)$ and $S_2(\mathbf{e}_2, \delta_y)$.

f_2 is locally injective. The following lemma summarizes the relationship of $S_2(\mathbf{e}_1, \delta_x)$ and $S_2(\mathbf{e}_2, \delta_y)$ with the local injectivity of function f_2.

LEMMA 1. *Function f_2 is locally injective all over the domain if no line simultaneously passes through the origin, $S_2(\mathbf{e}_1, \delta_x)$, and $S_2(\mathbf{e}_2, \delta_y)$.*

To determine the values of δ_x and δ_y that satisfy the condition in Lemma 1, it is necessary to represent the shape of $S_2(\mathbf{c}, \delta)$ in terms of $\mathbf{c}$ and δ. Since it is not easy to analyze the exact shape of $S_2(\mathbf{c}, \delta)$, we find two simple regions in R^2 that include $S_2(\mathbf{c}, \delta)$. The shape of $S_2(\mathbf{c}, \delta)$ can then be approximated by the intersection of the two regions.

3.2. *Bounding Regions of $S_2(\mathbf{c}, \delta)$*

Let $BS(\mathbf{c}, \delta)$ be a rectangular region in R^2 defined by

$$BS(\mathbf{c}, \delta) = \left\{ (x + c_x, y + c_y) \mid |x| \le \frac{3}{2}\delta, |y| \le \frac{3}{2}\delta \right\},$$

where $\mathbf{c} = (c_x, c_y)$. Let $\mathbf{O}$ denote the origin in R^2. Figure 4a shows the configuration of $BS(\mathbf{O}, \delta)$. It is simple to verify that $BS(\mathbf{c}, \delta)$ is a bounding region of $S_2(\mathbf{c}, \delta)$ by using the properties of uniform cubic B-spline basis functions. In the proofs of the following lemmas about bounding regions, without loss of generality, we only consider the case when $\mathbf{c}$ is the origin $\mathbf{O}$.

LEMMA 2. $S_2(\mathbf{c}, \delta) \subset BS(\mathbf{c}, \delta)$.

Proof. For each point $(x, y) \in S_2(\mathbf{O}, \delta)$, we have

$$\begin{aligned} |x| &= \left| \sum_{i=0}^{3} \sum_{j=0}^{3} D_{ij}^{u}(u, v)\delta_{ij} \right| \\ &\le \sum_{i=0}^{3} |B_i'(u)| \sum_{j=0}^{3} B_j(v)\delta \\ &\le \frac{3}{2}\delta. \end{aligned}$$

Similarly, we can show that $|y| \le \frac{3}{2}\delta$. ■

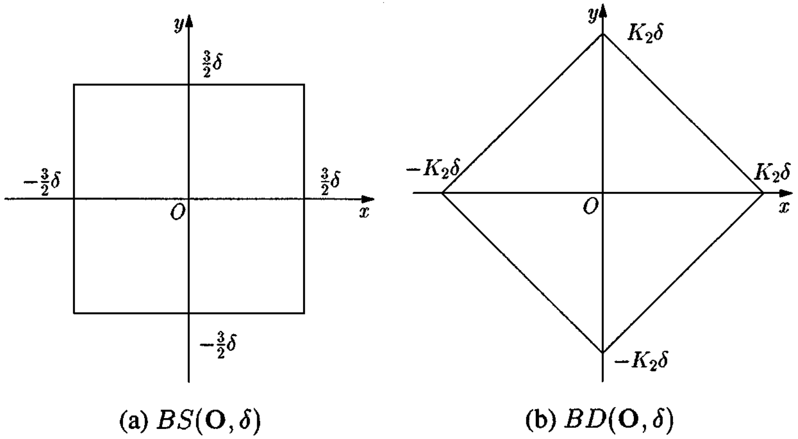

FIG. 4. Regions (a) $BS(\mathbf{O}, \delta)$ and (b) $BD(\mathbf{O}, \delta)$.

To obtain the second bounding region of $S_2(\mathbf{c}, \delta)$, we introduce a constant K_2 that is related to uniform cubic B-spline basis functions,

$$K_2 = \max_{0 \le u, v \le 1} \{g_2(u, v)\},$$

where

$$g_2(u, v) = \sum_{i=0}^{3} \sum_{j=0}^{3} \left| D_{ij}^u(u, v) + D_{ij}^v(u, v) \right|.$$

To compute the value of K_2, the domain $\Omega_2 = \{(u, v) \mid 0 \le u, v \le 1\}$ is partitioned to a very dense grid with the grid spacing 10^{-10}. We then evaluate the function g_2 at every grid point, and the maximum occurs at (u_0, u_0), where $u_0 = 0.2448210078$ or $u_0 = 0.7551789922$. Thus, K_2 is approximately 2.046392675.

Let $BD(\mathbf{c}, \delta)$ be a region in R^2 defined by

$$BD(\mathbf{c}, \delta) = \{(x + c_x, y + c_y) \mid |x + y| \le K_2 \delta, |x - y| \le K_2 \delta\}.$$

Figure 4b shows the configuration of $BD(\mathbf{O}, \delta)$. The following lemma shows that $BD(\mathbf{c}, \delta)$ is a bounding region of $S_2(\mathbf{c}, \delta)$.

LEMMA 3. $S_2(\mathbf{c}, \delta) \subset BD(\mathbf{c}, \delta)$.

Proof. Let $R_z(S_2(\mathbf{O}, \delta))$ be the region in R^2 that corresponds to the rotation of $S_2(\mathbf{O}, \delta)$ by $-\frac{\pi}{4}$ with respect to the origin,

$$R_z(S_2(\mathbf{O}, \delta)) = \left\{ (x', y') \mid x' = \frac{1}{\sqrt{2}}(x + y), y' = \frac{1}{\sqrt{2}}(-x + y) \right\},$$

where $(x, y) \in S_2(\mathbf{O}, \delta)$. Let $R_z(BD(\mathbf{O}, \delta))$ be the rotation of $BD(\mathbf{O}, \delta)$ by $-\frac{\pi}{4}$ with respect to the origin,

$$R_z(BD(\mathbf{O}, \delta)) = \left\{ (x', y') \mid |x'| \le \frac{K_2}{\sqrt{2}}\delta, |y'| \le \frac{K_2}{\sqrt{2}}\delta \right\}.$$

For each point $(x', y') \in R_z(S_2(\mathbf{O}, \delta))$, we have

$$\begin{aligned} |x'| &= \left| \frac{1}{\sqrt{2}} \sum_{i=0}^{3} \sum_{j=0}^{3} \left(D_{ij}^u(u, v) + D_{ij}^v(u, v) \right) \delta_{ij} \right| \\ &\le \frac{1}{\sqrt{2}} \sum_{i=0}^{3} \sum_{j=0}^{3} \left| D_{ij}^u(u, v) + D_{ij}^v(u, v) \right| |\delta_{ij}| \\ &\le \frac{K_2}{\sqrt{2}} \delta. \end{aligned}$$

Similarly, we can show that $|y'| \le \frac{K_2}{\sqrt{2}} \delta$ by using

$$K_2 = \max_{0 \le u, v \le 1} \left\{ \sum_{i=0}^{3} \sum_{j=0}^{3} \left| D_{ij}^u(u, v) - D_{ij}^v(u, v) \right| \right\},$$

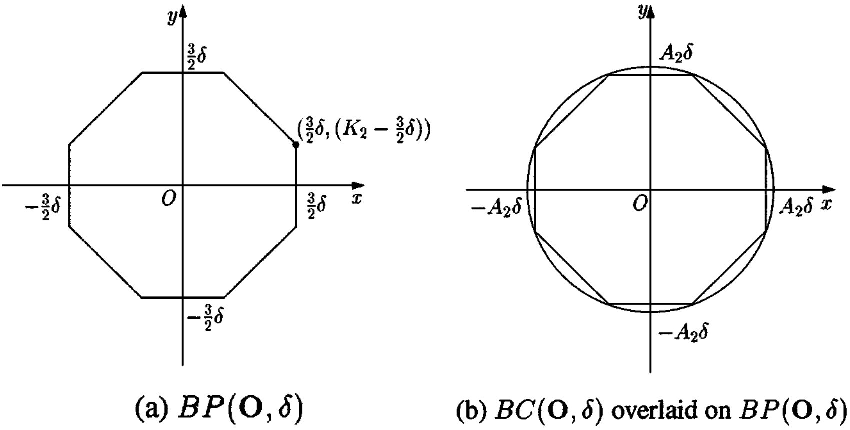

FIG. 5. Regions (a) $BP(\mathbf{O}, \delta)$ and (b) $BC(\mathbf{O}, \delta)$.

which holds from $B_j(v) = B_{3-j}(1-v)$ and $B'_j(v) = -B'_{3-j}(1-v)$. Then we have $R_z(S_2(\mathbf{O}, \delta)) \subset R_z(BD(\mathbf{O}, \delta))$, which implies $S_2(\mathbf{O}, \delta) \subset BD(\mathbf{O}, \delta)$. ■

Let $BP(\mathbf{c}, \delta)$ be the intersection of $BS(\mathbf{c}, \delta)$ and $BD(\mathbf{c}, \delta)$:

$$BP(\mathbf{c}, \delta) = BS(\mathbf{c}, \delta) \cap BD(\mathbf{c}, \delta).$$

Figure 5a shows the configuration of $BP(\mathbf{O}, \delta)$. It trivially holds that $BP(\mathbf{c}, \delta)$ is a polygonal bounding region of $S_2(\mathbf{c}, \delta)$.

LEMMA 4. $S_2(\mathbf{c}, \delta) \subset BP(\mathbf{c}, \delta)$.

Let $BC(\mathbf{c}, \delta)$ be a circular region in R^2 defined by

$$BC(\mathbf{c}, \delta) = \{(x + c_x, y + c_y) \mid x^2 + y^2 \leq (A_2\delta)^2\},$$

where $\mathbf{c} = (c_x, c_y)$ and

$$A_2 = \sqrt{\left(\frac{3}{2}\right)^2 + \left(K_2 - \frac{3}{2}\right)^2}.$$

$BC(\mathbf{c}, \delta)$ is the smallest circular region that encloses $BP(\mathbf{c}, \delta)$. Figure 5b shows the configuration of $BC(\mathbf{O}, \delta)$ overlaid on $BP(\mathbf{O}, \delta)$. It is obvious that $BC(\mathbf{c}, \delta)$ is a circular bounding region of $S_2(\mathbf{c}, \delta)$.

LEMMA 5. $S_2(\mathbf{c}, \delta) \subset BC(\mathbf{c}, \delta)$.

3.3. *Sufficient Conditions for 2D Injectivity*

With the geometric interpretation presented in Section 3.1 and the bounding regions obtained in Section 3.2, we can derive sufficient conditions for the local injectivity of a 2D uniform cubic B-spline function. Let f_2 be the function defined by Eq. (1) and $\Delta\phi_{ij} = \phi_{ij} - \phi^0_{ij} = (\Delta x_{ij}, \Delta y_{ij})$ for $i, j = 0, 1, 2, 3$. Let $\delta_x = \max\{|\Delta x_{ij}|\}$ and $\delta_y = \max\{|\Delta y_{ij}|\}$.

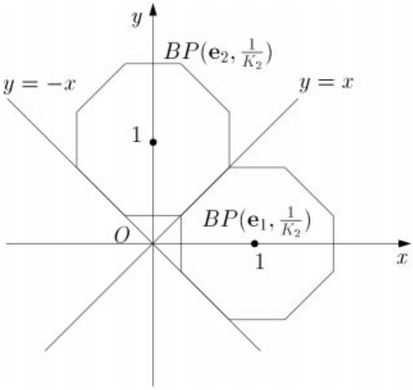

FIG. 6. The configuration of $BP(\mathbf{e}_1, \frac{1}{K_2})$ and $BP(\mathbf{e}_2, \frac{1}{K_2})$.

THEOREM 1. *Function f_2 is locally injective all over the domain if $\delta_x < \frac{1}{K_2}$ and $\delta_y < \frac{1}{K_2}$.*

Proof. Suppose that $\delta_x < \frac{1}{K_2}$ and $\delta_y < \frac{1}{K_2}$. Then we have $BP(\mathbf{e}_1, \delta_x) \subset \{(x, y) \mid x + y > 0, x - y > 0\}$ and $BP(\mathbf{e}_2, \delta_y) \subset \{(x, y) \mid x + y > 0, x - y < 0\}$ (see Fig. 6). Hence, no line that passes through the origin can intersect both $BP(\mathbf{e}_1, \delta_x)$ and $BP(\mathbf{e}_2, \delta_y)$. From Lemma 4, this implies that no line simultaneously passes through the origin, $S_2(\mathbf{e}_1, \delta_x)$, and $S_2(\mathbf{e}_2, \delta_y)$. Then, from Lemma 1, function f_2 is locally injective all over the domain. ■

In Theorem 1, the sufficient condition for local injectivity provides the same bound for the x- and y-displacements of control points. This condition cannot cover cases in which function f_2 may be locally injective if δ_y is much smaller than $\frac{1}{K_2}$ though $\delta_x \geq \frac{1}{K_2}$. We can obtain a sufficient condition that can handle such a case by using the circular bounding region $BC(\mathbf{c}, \delta)$. We first show a lemma related to a line passing through the origin and regions $BC(\mathbf{e}_1, \delta_x)$ and $BC(\mathbf{e}_2, \delta_y)$.

LEMMA 6. *If there is a line that simultaneously passes through the origin, $BC(\mathbf{e}_1, \delta_x)$, and $BC(\mathbf{e}_2, \delta_y)$, then $\delta_x^2 + \delta_y^2 \geq (\frac{1}{A_2})^2$.*

Proof. Let l be a line that passes through the origin, which is represented by $ax + by = 0$. Let d_1 and d_2 be the distances of $\mathbf{e}_1$ and $\mathbf{e}_2$ from line l, respectively. Then we have $d_1 = |a|/\sqrt{a^2 + b^2}$ and $d_2 = |b|/\sqrt{a^2 + b^2}$, which gives $d_1^2 + d_2^2 = 1$. Suppose that line l simultaneously intersects $BC(\mathbf{e}_1, \delta_x)$ and $BC(\mathbf{e}_2, \delta_y)$. Figure 7 shows an example of such a

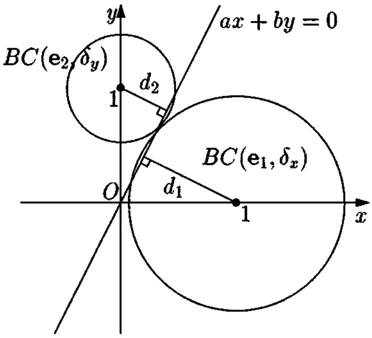

FIG. 7. A configuration of $BC(\mathbf{e}_1, \delta_x)$ and $BC(\mathbf{e}_2, \delta_y)$ with a line passing through the origin.

configuration. Then we have $d_1 \leq A_2\delta_x$ and $d_2 \leq A_2\delta_y$. This implies that $1 = d_1^2 + d_2^2 \leq A_2^2(\delta_x^2 + \delta_y^2)$. ■

THEOREM 2. *Function f_2 is locally injective all over the domain if $\delta_x^2 + \delta_y^2 < (\frac{1}{A_2})^2$.*

Proof. Suppose that $\delta_x^2 + \delta_y^2 < (\frac{1}{A_2})^2$. Then, from Lemma 6, there cannot be a line passing through the origin that simultaneously intersects $BC(\mathbf{e}_1, \delta_x)$ and $BC(\mathbf{e}_2, \delta_y)$. This implies from Lemmas 1 and 5 that function f_2 is locally injective all over the domain. ■

Since the constant K_2 is approximately 2.046392675, the bounds $\frac{1}{K_2}$ and $(\frac{1}{A_2})^2$ given in Theorems 1 and 2 are approximately 0.488664767 and 0.392380757, respectively. Thus it seems that the sufficient conditions in Theorems 1 and 2 only provide very small bounds for control point displacements. However, by using the affine invariance property of B-splines, we can extend Theorems 1 and 2 to handle general cases with possibly large control point displacements. That is, we can apply the bounds to the normalized values of control point displacements instead of the given values.

To normalize the control point displacements, we consider a 2D affine transformation M_2 that moves the given control points ϕ_{ij} toward the canonical positions ϕ_{ij}^0. Let ϕ'_{ij} be the new position of a control point ϕ_{ij} when it is transformed by M_2. We determine the affine transformation M_2 so that it minimizes the approximation error, $\sum_{i=0}^{3}\sum_{j=0}^{3} \|\phi'_{ij} - \phi_{ij}^0\|^2$. Such a transformation M_2 can simply be obtained by using the matrix representation of M_2. For details of inferring an affine transformation from a set of corresponding point pairs and the least-squares solution of a linear system, refer to the references [11, 14]. Let $\Delta\phi'_{ij} = \phi'_{ij} - \phi_{ij}^0 = (\Delta x'_{ij}, \Delta y'_{ij})$. Let $\delta'_x = \max\{|\Delta x'_{ij}|\}$ and $\delta'_y = \max\{|\Delta y'_{ij}|\}$.

THEOREM 3. *Function f_2 is locally injective all over the domain if the 2D affine transformation M_2 is invertible and if δ'_x and δ'_y satisfy one of the conditions $\delta'_x < \frac{1}{K_2}$ and $\delta'_y < \frac{1}{K_2}$ or $(\delta'_x)^2 + (\delta'_y)^2 < (\frac{1}{A_2})^2$.*

Proof. Let f'_2 be the 2D uniform cubic B-spline function determined by transformed control points ϕ'_{ij}. If $\delta'_x < \frac{1}{K_2}$ and $\delta'_y < \frac{1}{K_2}$ or if $(\delta'_x)^2 + (\delta'_y)^2 < (\frac{1}{A_2})^2$, function f'_2 is locally injective all over the domain from Theorem 1 or Theorem 2, respectively. Due to the affine invariance property of B-splines, the function value $f_2(u, v)$ is mapped to the function value $f'_2(u, v)$ by transformation M_2. If M_2 is invertible, the mapping between $f_2(u, v)$ and $f'_2(u, v)$ is a one-to-one correspondence. ■

For example, when control point displacements $\Delta\phi_{ij}$ are (5, 5) for all i, j, it is obvious that function f_2 is locally injective all over the domain. However, Theorems 1 and 2 do not cover this case because the conditions in them are not satisfied by $\Delta\phi_{ij} = (5, 5)$. In contrast, when we normalize the displacements $\Delta\phi_{ij}$, transformation M_2 is a translation by $(-5, -5)$ and $\Delta\phi'_{ij} = (0, 0)$ for all i, j, which obviously satisfies the condition in Theorem 3.

The sufficient conditions in Theorems 1 and 2 are not necessary for the local injectivity of function f_2 even if we apply them to the normalized control point displacements. For example, let $\Delta x_{0j} = \Delta x_{1j} = -\Delta x_{2j} = -\Delta x_{3j} = 0.65$ and $\Delta y_{ij} = 0$, for $i, j = 0, 1, 2, 3$. In this case, both of the conditions in Theorems 1 and 2 are violated but function f_2 is still locally injective.

However, the bound for the control point displacements in Theorem 1 is tight. Lee *et al.* [9] presented a control lattice configuration with which function f_2 is not locally injective when $\delta_x = \delta_y = \frac{1}{K_2}$. In the configuration, $\Delta x_{ij} = \Delta y_{ij} = \frac{1}{K_2}$ if $i + j \leq 3$ and $\Delta x_{ij} = \Delta y_{ij} = -\frac{1}{K_2}$ if $i + j > 3$ for $i, j = 0, 1, 2, 3$. Then the Jacobian $|J(f_2)|$ vanishes for the point (u_0, u_0) at which the function g_2 has the maximum value K_2. For Theorem 2, we have not yet fully investigated the tightness of the inequality in it.

Theorem 2 is more useful than Theorem 1 when δ_x and δ_y are different. For example, given a control lattice with $\delta_x = 0.62$ and $\delta_y = 0$, we cannot verify the local injectivity of the resulting function f_2 with Theorem 1. In contrast, Theorem 2 can be used to guarantee that the function f_2 is locally injective.

4. INJECTIVITY CONDITIONS OF 3D B-SPLINE FUNCTIONS

In this section, we present sufficient conditions for the local injectivity of a 3D uniform cubic B-spline function by expanding the geometric interpretation and the bounding regions presented in Section 3 to 3D space.

4.1. *Geometric Interpretation of 3D Injectivity*

Let f_3 be a 3D uniform cubic B-spline function defined by Eq. (2). When ϕ_{ijk} equals $\phi^0_{ijk} = (i-1, j-1, k-1)$ for $i, j, k = 0, 1, 2, 3$, function f_3 is reduced to an identity function. Let $\Delta\phi_{ijk}$ be the displacement of control point ϕ_{ijk} from ϕ^0_{ijk}; that is, $\Delta\phi_{ijk} = \phi_{ijk} - \phi^0_{ijk} = (\Delta x_{ijk}, \Delta y_{ijk}, \Delta z_{ijk})$. Then we have

$$f_3(u, v, w) = (u, v, w) + \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} B_i(u)B_j(v)B_k(w)\Delta\phi_{ijk}.$$

Let $\mathbf{r}_1 = (\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial x}{\partial w})$, $\mathbf{r}_2 = (\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial y}{\partial w})$, and $\mathbf{r}_3 = (\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w})$ denote the row vectors of the Jacobian matrix $J(f_3)$. Let $D^u_{ijk}(u, v, w) = B'_i(u)B_j(v)B_k(w)$, $D^v_{ijk}(u, v, w) = B_i(u)B'_j(v)B_k(w)$, and $D^w_{ijk}(u, v, w) = B_i(u)B_j(v)B'_k(w)$. We define $S_3(\mathbf{c}, \delta)$ as a region in R^3 such that

$$S_3(\mathbf{c}, \delta) = \Bigg\{ (x + c_x, y + c_y, z + c_z) \mid x = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} D^u_{ijk}(u, v, w)\delta_{ijk}, \\ y = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} D^v_{ijk}(u, v, w)\delta_{ijk}, z = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} D^w_{ijk}(u, v, w)\delta_{ijk} \Bigg\},$$

where $\mathbf{c} = (c_x, c_y, c_z)$, $|\delta_{ijk}| \le \delta$, and $0 \le u, v, w \le 1$. $S_3(\mathbf{c}, \delta)$ is the 3D expansion of the region $S_2(\mathbf{c}, \delta)$. Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$. Figure 8 shows a schematic diagram of the regions $S_3(\mathbf{e}_1, \delta_x)$, $S_3(\mathbf{e}_2, \delta_y)$, and $S_3(\mathbf{e}_3, \delta_z)$.

Let $\delta_x = \max\{|\Delta x_{ijk}|\}$, $\delta_y = \max\{|\Delta y_{ijk}|\}$, and $\delta_z = \max\{|\Delta z_{ijk}|\}$. The 3D extension of Lemma 1 can easily be proved from the fact that three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ in R^3 are linearly independent if and only if there is no plane passing through the origin that contains the points $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. As in the 2D case, we interchangeably use a 3D vector and a point in R^3.

LEMMA 7. *Function f_3 is locally injective all over the domain if no plane simultaneously passes through the origin, $S_3(\mathbf{e}_1, \delta_x)$, $S_3(\mathbf{e}_2, \delta_y)$, and $S_3(\mathbf{e}_3, \delta_z)$.*

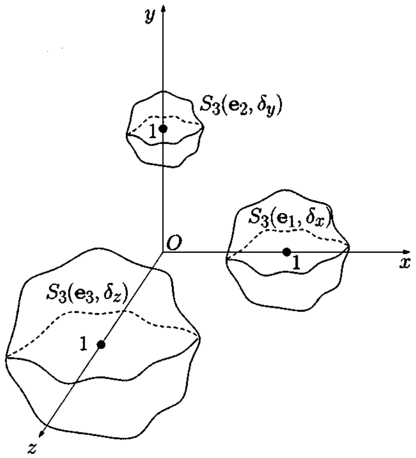

FIG. 8. A schematic diagram of $S_3(\mathbf{e}_1, \delta_x)$, $S_3(\mathbf{e}_2, \delta_y)$, and $S_3(\mathbf{e}_3, \delta_z)$.

4.2. *Bounding Regions of* $S_3(\mathbf{c}, \delta)$

Let $BTS(\mathbf{c}, \delta)$ be a region in R^3 defined by

$$BTS(\mathbf{c}, \delta) = \left\{(x + c_x, y + c_y, z + c_z) \mid |x| \leq \frac{3}{2}\delta, |y| \leq \frac{3}{2}\delta, |z| \leq \frac{3}{2}\delta\right\},$$

where $\mathbf{c} = (c_x, c_y, c_z)$. It can be proved in a similar way to Lemma 2 that $BTS(\mathbf{c}, \delta)$ is a bounding region of $S_3(\mathbf{c}, \delta)$.

LEMMA 8. $S_3(\mathbf{c}, \delta) \subset BTS(\mathbf{c}, \delta)$.

Let $BTD(\mathbf{c}, \delta)$ be a region in R^3 defined by

$$BTD(\mathbf{c}, \delta) = BTX(\mathbf{c}, \delta) \cap BTY(\mathbf{c}, \delta) \cap BTZ(\mathbf{c}, \delta),$$

where

$$BTX(\mathbf{c}, \delta) = \left\{(x + c_x, y + c_y, z + c_z) \mid |x| \leq \frac{3}{2}\delta, |y + z| \leq K_2\delta, |y - z| \leq K_2\delta\right\},$$

$$BTY(\mathbf{c}, \delta) = \left\{(x + c_x, y + c_y, z + c_z) \mid |y| \leq \frac{3}{2}\delta, |z + x| \leq K_2\delta, |z - x| \leq K_2\delta\right\},$$

$$BTZ(\mathbf{c}, \delta) = \left\{(x + c_x, y + c_y, z + c_z) \mid |z| \leq \frac{3}{2}\delta, |x + y| \leq K_2\delta, |x - y| \leq K_2\delta\right\},$$

and $\mathbf{c} = (c_x, c_y, c_z)$. Note that $BTZ(\mathbf{c}, \delta)$ is an extrusion of the 2D region $BD(\mathbf{c}, \delta)$ in the z-direction. Similarly, $BTX(\mathbf{c}, \delta)$ and $BTY(\mathbf{c}, \delta)$ are extrusions of $BD(\mathbf{c}, \delta)$ when it is defined in the yz- and zx-planes, respectively. It is simple to show that $BTD(\mathbf{c}, \delta)$ is a bounding region of $S_3(\mathbf{c}, \delta)$.

LEMMA 9. $S_3(\mathbf{c}, \delta) \subset BTD(\mathbf{c}, \delta)$.

Proof. It can be shown that $S_3(\mathbf{O}, \delta) \subset BTZ(\mathbf{O}, \delta)$ in a way similar to Lemma 3. Also, we can show that $S_3(\mathbf{O}, \delta) \subset BTX(\mathbf{O}, \delta)$ and $S_3(\mathbf{O}, \delta) \subset BTY(\mathbf{O}, \delta)$ by using rotations with respect to the x- and y-axes, respectively. ■

To obtain a regular octahedron that includes $S_3(\mathbf{c}, \delta)$, we introduce a constant K_3, which is the 3D correspondence of K_2,

$$K_3 = \max_{0 \le u,v,w \le 1} \{g_3(u, v, w)\},$$

where

$$g_3(u, v, w) = \sum_{i=0}^{3} \sum_{j=0}^{3} \sum_{k=0}^{3} \left| D_{ijk}^{u}(u, v, w) + D_{ijk}^{v}(u, v, w) + D_{ijk}^{w}(u, v, w) \right|.$$

To compute the value of K_3, the domain, $\Omega_3 = \{(u, v, w) \mid 0 \le u, v, w \le 1\}$, is partitioned into a very dense grid with the grid spacing 10^{-10}. We then evaluate the function g_3 at every grid point, and the maximum occurs at (u_0, u_0, u_0), where $u_0 = 0.1640662347$ or $u_0 = 0.8359337653$. Thus, K_3 is approximately 2.479472335.

Let $BTO(\mathbf{c}, \delta)$ be a region in R^3 defined by

$$BTO(\mathbf{c}, \delta) = BTP_1(\mathbf{c}, \delta) \cap BTP_2(\mathbf{c}, \delta) \cap BTP_3(\mathbf{c}, \delta) \cap BTP_4(\mathbf{c}, \delta),$$

where

$$\begin{aligned}
BTP_1(\mathbf{c}, \delta) &= \{(x + c_x, y + c_y, z + c_z) \mid |x + y + z| \le K_3\delta\},\\
BTP_2(\mathbf{c}, \delta) &= \{(x + c_x, y + c_y, z + c_z) \mid |x + y - z| \le K_3\delta\},\\
BTP_3(\mathbf{c}, \delta) &= \{(x + c_x, y + c_y, z + c_z) \mid |x - y + z| \le K_3\delta\},\\
BTP_4(\mathbf{c}, \delta) &= \{(x + c_x, y + c_y, z + c_z) \mid |x - y - z| \le K_3\delta\}
\end{aligned}$$

and $\mathbf{c} = (c_x, c_y, c_z)$. The following lemma shows that $BTO(\mathbf{c}, \delta)$ is a bounding region of $S_3(\mathbf{c}, \delta)$.

LEMMA 10. $S_3(\mathbf{c}, \delta) \subset BTO(\mathbf{c}, \delta)$.

Proof. Consider a 3D rotation R_{xy} that maps the vector $(1, 1, 1)$ onto the z-axis. Rotation R_{xy} transforms $BTP_1(\mathbf{O}, \delta)$ to $R_{xy}(BTP_1(\mathbf{O}, \delta))$, where

$$R_{xy}(BTP_1(\mathbf{O}, \delta)) = \left\{(x', y', z') \mid |z'| \le \frac{K_3}{\sqrt{3}}\delta\right\}.$$

Let z' be the z-coordinate of the new position when we apply rotation R_{xy} to a point $(x, y, z) \in S_3(\mathbf{O}, \delta)$. Then we have

$$\begin{aligned}
|z'| &= \left|\frac{1}{\sqrt{3}}(x + y + z)\right|\\
&\le \frac{1}{\sqrt{3}} \sum_{i=0}^{3} \sum_{j=0}^{3} \sum_{k=0}^{3} \left| D_{ijk}^{u}(u, v, w) + D_{ijk}^{v}(u, v, w) + D_{ijk}^{w}(u, v, w) \right| |\delta_{ijk}|\\
&\le \frac{K_3}{\sqrt{3}}\delta,
\end{aligned}$$

which implies $S_3(\mathbf{O}, \delta) \subset BTP_1(\mathbf{O}, \delta)$. Similarly, we can show that $S_3(\mathbf{O}, \delta) \subset BTP_2(\mathbf{O}, \delta)$ $S_3(\mathbf{O}, \delta) \subset BTP_3(\mathbf{O}, \delta)$, and $S_3(\mathbf{O}, \delta) \subset BTP_4(\mathbf{O}, \delta)$. ■

Let $BPH(\mathbf{c}, \delta)$ be the intersection of $BTS(\mathbf{c}, \delta)$, $BTD(\mathbf{c}, \delta)$, and $BTO(\mathbf{c}, \delta)$:

$$BPH(\mathbf{c}, \delta) = BTS(\mathbf{c}, \delta) \cap BTD(\mathbf{c}, \delta) \cap BTO(\mathbf{c}, \delta).$$

The boundary of $BPH(\mathbf{c}, \delta)$ consists of 26 faces. Figure 9 illustrates the shape of $BPH(\mathbf{O}, 1)$. Figure 9a is the shape of $BPH(\mathbf{O}, 1)$ that is seen from the x-axis toward the origin. Note that, due to the symmetry, the shape is the same when $BPH(\mathbf{O}, 1)$ is seen from the y- or z-axis. The shape in Fig. 9b is obtained when we look at $BPH(\mathbf{O}, 1)$ from the line $y = x$ in the xy-plane toward the origin. Figure 9c shows the shape of $BPH(\mathbf{O}, 1)$ with a viewpoint looking downward. In Fig. 9, magenta, yellow, and cyan faces are originally part of the faces of $BTS(\mathbf{O}, 1)$, $BTD(\mathbf{O}, 1)$, and $BTO(\mathbf{O}, 1)$, respectively. It trivially holds that $BPH(\mathbf{c}, \delta)$ is a polyhedral bounding region of $S_3(\mathbf{c}, \delta)$.

LEMMA 11. $S_3(\mathbf{c}, \delta) \subset BPH(\mathbf{c}, \delta)$.

Let $BSP(\mathbf{c}, \delta)$ be a spherical region defined by

$$BSP(\mathbf{c}, \delta) = \{(x + c_x, y + c_y, z + c_z) \mid x^2 + y^2 + z^2 \leq (A_3\delta)^2\},$$

where $\mathbf{c} = (c_x, c_y, c_z)$ and

$$A_3 = \sqrt{\left(\frac{3}{2}\right)^2 + \left(K_2 - \frac{3}{2}\right)^2 + (K_3 - K_2)^2}.$$

$BSP(\mathbf{c}, \delta)$ is the smallest spherical region that encloses $BPH(\mathbf{c}, \delta)$. It is obvious that $BSP(\mathbf{c}, \delta)$ is a bounding region of $S_3(\mathbf{c}, \delta)$.

LEMMA 12. $S_3(\mathbf{c}, \delta) \subset BSP(\mathbf{c}, \delta)$.

4.3. *Sufficient Conditions for 3D Injectivity*

Now we present sufficient conditions for the local injectivity of a 3D uniform cubic B-spline function. Let f_3 be the function defined by Eq. (2) and let $\Delta\phi_{ijk} = \phi_{ijk} - \phi^0_{ijk} = (\Delta x_{ijk}, \Delta y_{ijk}, \Delta z_{ijk})$ for $i, j, k = 0, 1, 2, 3$. Let $\delta_x = \max\{|\Delta x_{ijk}|\}$, $\delta_y = \max\{|\Delta y_{ijk}|\}$, and $\delta_z = \max\{|\Delta z_{ijk}|\}$.

THEOREM 4. *Function f_3 is locally injective all over the domain if $\delta_x < \frac{1}{K_3}$, $\delta_y < \frac{1}{K_3}$, and $\delta_z < \frac{1}{K_3}$.*

Proof. By using the polyhedral bounding region $BPH(\mathbf{c}, \delta)$ of $S_3(\mathbf{c}, \delta)$, this theorem can be proved in a way similar to Theorem 1. ■

THEOREM 5. *Function f_3 is locally injective all over the domain if $\delta_x^2 + \delta_y^2 + \delta_z^2 < (\frac{1}{A_3})^2$.*

Proof. Using the spherical bounding region $BSP(\mathbf{c}, \delta)$ of $S_3(\mathbf{c}, \delta)$, this theorem can be proved in a way similar to Theorem 2. ■

To normalize the control point displacements, we consider a 3D affine transformation M_3. Let ϕ'_{ijk} be the new position of a control point ϕ_{ijk} when it is transformed by

M_3. We determine the affine transformation M_3 so that it minimizes the approximation error, $\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3}\|\phi'_{ijk}-\phi^0_{ijk}\|^2$. Let $\Delta\phi'_{ijk}=\phi'_{ijk}-\phi^0_{ijk}=(\Delta x'_{ijk},\Delta y'_{ijk},\Delta z'_{ijk})$. Let $\delta'_x=\max\{|\Delta x'_{ijk}|\}$, $\delta'_y=\max\{|\Delta y'_{ijk}|\}$, and $\delta'_z=\max\{|\Delta z'_{ijk}|\}$.

THEOREM 6. *Function f_3 is locally injective all over the domain if the 3D affine transformation M_3 is invertible and if δ'_x, δ'_y, and δ'_z satisfy one of the conditions $\delta'_x<\frac{1}{K_3}$, $\delta'_y<\frac{1}{K_3}$, and $\delta'_z<\frac{1}{K_3}$ or $(\delta'_x)^2+(\delta'_y)^2+(\delta'_z)^2<(\frac{1}{A_3})^2$.*

Proof. By using the affine invariance property of B-splines, this theorem can be proved in a way similar to Theorem 3. ■

As in the 2D case, the sufficient conditions in Theorems 4 and 5 are not necessary for the local injectivity of function f_3. However, similarly to Theorem 1, Theorem 4 provides a tight bound for the control point displacements. There is a control lattice configuration with which f_3 is not locally injective when $\delta_x=\delta_y=\delta_z=\frac{1}{K_3}$. In the configuration, $\Delta x_{ijk}=\Delta y_{ijk}=\Delta z_{ijk}=\frac{1}{K_3}$ if $i+j+k\leq 3$ and $\Delta x_{ij}=\Delta y_{ij}=\Delta z_{ijk}=-\frac{1}{K_2}$ if $i+j+k>3$ for $i,j,k=0,1,2,3$. Then the Jacobian $|J(f_3)|$ vanishes for the point (u_0,u_0,u_0) at which the function g_3 has the maximum value K_3. We have not yet fully investigated the tightness of the inequality in Theorem 5.

5. CONCLUSIONS

In this paper, we have presented sufficient conditions for the local injectivity of 2D and 3D uniform cubic B-spline functions. We first proposed a geometric interpretation of the local injectivity of a uniform cubic B-spline function. Row vectors in the Jacobian matrix are mapped to regions in the space, for which we obtained simple bounding regions. Sufficient conditions represented in terms of control point displacements were finally derived by using the bounding regions. These sufficient conditions are straightforward and easily tested so that they may be useful to guarantee the injectivity of mapping functions in applications such as image warping and morphing, 3D deformation, and volume morphing.

Future work will involve an investigation of sufficient conditions that consider each control point displacement independently. In this paper, sufficient conditions are represented only in terms of the maximums of the control point displacements in principal directions. The ultimate future goal will be to obtain simple and easy-to-check necessary and sufficient conditions for the injectivity of uniform cubic B-spline functions.

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